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INJECTIVITY OVER THE COENDOMORPHISM CORING OF A QUASI-FINITE COMODULE

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Abstract

Let k be a field, A and B two algebras over k with B a QF ring. Let $\mathcal{C}$ be an A-coring that is flat as a left A-module. Let Y be a $(B,\mathcal{C})$-quasi-finite object of the category ${}_B\mathcal{M}^{\mathcal{C}}$ of $(B,\mathcal{C})$-bicomodules. Assume that the coendomorphism coring $\mathcal{D}$ of Y is projective as a left B-module. We give necessary and sufficient conditions for an object of the category $\mathcal{M}^{\mathcal{D}}$ of right $\mathcal{D}$-comodules to be injective in $\mathcal{M}^{\mathcal{D}}$. If the algebra B is a division ring, then $\mathcal{D}$ is projective as a left B-module and our result can be applied.

Keywords: Coring, coalgebra, comodule over a coring, comodule over a coalgebra, coendomorphism coring, quasi-finite comodule, injectivity, coflatness.

1 Introduction

Comodules over a coring were introduced in [1] such that entwined modules (hence Doi-Hopf modules) are special cases of comodules over a certain coring. After 2000 the study of comodules over a coring generated an explosion of interest soon after the monographs [3] and [5] were published.

Let k be a field, A a k-algebra and $\mathcal{C}$ an A-coring. The left A-linear maps $\mathcal{C} \to A$ have a ring structure. This allows one to define a dual ring ${}^*\mathcal{C}$. Let Λ be a right $\mathcal{C}$-comodule. In [8] when $\mathcal{C}$ is projective as a left A-module, we have given necessary and sufficient conditions for an object of $\mathcal{M}^{\mathcal{C}}$ to be projective (resp. flat) as a module over the endomorphism ring $End^{\mathcal{C}}(\Lambda)$ of Λ if Λ is finitely generated (resp. finitely presented) as a left ${}^*\mathcal{C}$-module. Let $\mathcal{C}$ be flat as a left A-module, B a k-algebra which is a QF ring, Y a $(B, \mathcal{C})$-quasi-finite object of the category ${}_B\mathcal{M}^{\mathcal{C}}$ of $(B, \mathcal{C})$-bicomodules and $\mathcal{D} = e_{\mathcal{C}}(Y)$ the coendomorphism coring of Y. Assume that $\mathcal{D}$ is projective as a left B-module. In the present paper, we give necessary and sufficient conditions for a right $\mathcal{D}$-comodule to be injective in $\mathcal{M}^{\mathcal{D}}$. If the algebra B is a division ring (for example, $B = k$), then $\mathcal{D}$ is projective as a left B-module and our result can be applied. If H is a Hopf algebra, C is a right H-comodule coalgebra and $C \bowtie H$ is the smash coproduct, then C is $C \bowtie H$-quasi-finite. Hence we get necessary and sufficient conditions for injectivity over the coendomorphism coring $e_{C\bowtie H}(C)$ of C.

2 Preliminary results

Let k be a field and A a k-algebra. An A-coring $\mathcal{C}$ is an (A, A)-bimodule together with two (A, A)-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that the usual coassociativity and counit properties hold. For more details on corings, we refer to [1], [2], [3] and [5]. Let $\mathcal{C}$ be an A-coring. A right $\mathcal{C}$-comodule is a right A-module M together with a right A-linear map $\rho_{M,\mathcal{C}} : M \to M \otimes_A \mathcal{C}; m \mapsto m_0 \otimes_A m_1$ such that

$$(id_M \otimes_A \epsilon_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}} = id_M, \quad \text{and} \quad (id_M \otimes_A \Delta_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}} = (\rho_{M,\mathcal{C}} \otimes_A id_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}}.$$

A morphism of right $\mathcal{C}$-comodules $f : M \to N$ is a right A-linear map such that

$$\rho_{N,\mathcal{C}} \circ f = (f \otimes_A id_M) \circ \rho_{M,\mathcal{C}}.$$

We denote the set of comodule morphisms between M and N by $Hom^{\mathcal{C}}(M, N)$.

We know that $\rho_{M,\mathcal{C}}$ is an injection of right $\mathcal{C}$-comodules: the $\mathcal{C}$-coaction on $M \otimes_A \mathcal{C}$ is given by $id_M \otimes_A \Delta_{\mathcal{C}}$. Therefore, every right $\mathcal{C}$-comodule M is a right $\mathcal{C}$-subcomodule of $M \otimes_A \mathcal{C}$.

Let us denote by $\mathcal{M}^{\mathcal{C}}$ the category formed by right $\mathcal{C}$-comodules and comodule morphisms and by $\mathcal{M}$ the category of k-vector spaces. By ([3], 18.8), the category $\mathcal{M}^{\mathcal{C}}$ has direct sums.

Let M be a right $\mathcal{C}$-comodule with structure map $\rho_{M,\mathcal{C}}$ and N a left $\mathcal{C}$-comodule $\rho_{\mathcal{C},N}$. Then the cotensor product $M \square_{\mathcal{C}} N$ is the k-vector space defined by

$$M \square_{\mathcal{C}} N = \{m \otimes_k n \in M \otimes_k N : \quad \rho_{M,\mathcal{C}}(m) \otimes_k n = m \otimes_k \rho_{\mathcal{C},N}(n)\}.$$

Let $f : M \to M'$ be a morphism of right $\mathcal{C}$-comodules and $g : N \to N'$ be a morphism of left $\mathcal{C}$-comodules. The cotensor product of f and g denoted $f \square g$ is the k-linear map defined from $M \square_{\mathcal{C}} N \to M' \square_{\mathcal{C}} N'$ by $(f \square g)(m \square n) = f(m) \square g(n)$ for all $m \in M$ and $n \in N$.

Let B be a k-algebra and $\mathcal{D}$ a B-coring. A $(\mathcal{D}, \mathcal{C})$-bicomodule M is a (B, A)-bimodule which is a right $\mathcal{C}$-comodule, a left $\mathcal{D}$-comodule with coactions $\rho_M : M \otimes_A \mathcal{C}$ and $\rho_{\mathcal{D},M} : M \to \mathcal{D} \otimes_B M$ such that the diagram

$$\begin{array}{ccc}
\mathbf{M} & \xrightarrow{\rho_{M,\mathcal{C}}} & \mathbf{M} \otimes_{\mathbf{A}} \mathcal{C} \\
\rho_{\mathcal{D},M} \Big\downarrow & & \Big\downarrow \rho_{\mathcal{D},M} \otimes I_{\mathcal{C}} \\
\mathcal{D} \otimes_{\mathbf{B}} \mathbf{M} & \xrightarrow[I_{\mathcal{D}} \otimes \rho_{M,\mathcal{C}}]{} & \mathcal{D} \otimes_{\mathbf{B}} \mathbf{M} \otimes_{\mathbf{A}} \mathcal{C}
\end{array}$$

is commutative, that is, $\rho_{M,\mathcal{C}}$ is a left $\mathcal{D}$-comodule morphism, or equivalently, $\rho_{\mathcal{D},M}$ is a right $\mathcal{C}$-comodule morphism. We denote by ${}^{\mathcal{D}}\mathcal{M}^{\mathcal{C}}$ the category of

$(\mathcal{D},\mathcal{C})$-bicomodules. When $\mathcal{D} = B$ is the trivial coring, then ${}^{\mathcal{D}}\mathcal{M}^{\mathcal{C}}$ is just the category ${}_B\mathcal{M}^{\mathcal{C}}$ of $(B,\mathcal{C})$-bicomodules that are (B,A)-bimodules such that the coaction map is a (B,A)-bimodule map. Morphisms of ${}_B\mathcal{M}^{\mathcal{C}}$ are left B-linear and right $\mathcal{C}$-colinear maps ([3], 39.2). An object Q of $\mathcal{M}^{\mathcal{C}}$ is injective in $\mathcal{M}^{\mathcal{C}}$ if, for any monomorphism $M \to N$ in $\mathcal{M}^{\mathcal{C}}$, the canonical map $Hom(N,Q) \to Hom(M,Q)$ is surjective.

An object M of $\mathcal{M}^{\mathcal{C}}$ is termed an A-relative injective comodule or a $(\mathcal{C},A)$-injective comodule provided that, for every right $\mathcal{C}$-comodule map $i : N \to L$ that is a coretraction in $\mathcal{M}_A$, every diagram

$$\begin{array}{ccc} \mathbf{M} & & \\ {\scriptstyle f}\big\uparrow & & \\ \mathbf{N} & \xrightarrow[i]{} & \mathbf{L} \end{array}$$

in $\mathcal{M}^{\mathcal{C}}$ can be completed commutatively by some $g : L \to M$ in $\mathcal{M}^{\mathcal{C}}$.

An object Y in ${}_B\mathcal{M}^{\mathcal{C}}$ is a $(B,\mathcal{C})$-injector if the tensor functor

$$- \otimes_B Y : \mathcal{M}_B \to \mathcal{M}^{\mathcal{C}}$$

respects injective objects. Let Y be an object of ${}_B\mathcal{M}^{\mathcal{C}}$ and W an object of $\mathcal{M}_B$. Then $W \otimes_B Y$ is an object of $\mathcal{M}^{\mathcal{C}}$.

An object Y in ${}_B\mathcal{M}^{\mathcal{C}}$ is $(B,\mathcal{C})$-quasi-finite if the tensor functor

$$- \otimes_B Y : \mathcal{M}_B \to \mathcal{M}^{\mathcal{C}}$$

has a left adjoint. This left adjoint is a covariant functor called the Cohom functor and is denoted

$$h_{\mathcal{C}}(Y,-) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_B.$$

Explicitly, this means that, for all $M \in \mathcal{M}^{\mathcal{C}}$ and $W \in \mathcal{M}_B$, there exists a functorial isomorphism

$$\Phi_{M,W} : Hom_B(h_{\mathcal{C}}(Y,M),W) \to Hom^{\mathcal{C}}(M, W \otimes_B Y).$$

Let $Y \in {}^{\mathcal{D}}\mathcal{M}^{\mathcal{C}}$ be $(B,\mathcal{C})$-quasi-finite and $M \in \mathcal{M}^{\mathcal{C}}$. By ([3], 23.5), there is a unique left $\mathcal{D}$-comodule structure on $h_{\mathcal{C}}(Y,M)$ and we get a functor

$$h_{\mathcal{C}}(Y,-) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}.$$

In the remainder of the paper we assume that $\mathcal{C}$ is flat as a left A-module.

By ([3], 18.14), $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category. Therefore it has enough injectives. Moreover, a right $\mathcal{C}$-comodule Q is injective if and only if the functor $Hom^{\mathcal{C}}(-,Q) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ is exact (see [3], page 187). A right $\mathcal{C}$-comodule Q is coflat if and only if the functor : $Q\Box_{\mathcal{C}}(-) : {}^{\mathcal{C}}\mathcal{M} \to \mathcal{M}$ is exact ([3], 21.6).

By the proof of ([3], 23.6), the functor $F = h_{\mathcal{C}}(Y,-)$ is a left adjoint to the functor

$$G = -\Box_{\mathcal{D}}Y : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}.$$

Then for $M \in \mathcal{M}^{\mathcal{C}}$ and $W \in \mathcal{M}^{\mathcal{D}}$, there exists a functorial isomorphism

$$\Phi_{M,W} : Hom^{\mathcal{D}}(h_{\mathcal{C}}(Y,M),W) \to Hom^{\mathcal{C}}(M, W\Box_{\mathcal{D}}Y).$$

The unit of the adjunction is given by

$$u_N : N \to h_{\mathcal{C}}(Y,N)\Box_{\mathcal{D}}Y \quad \text{for} \quad N \in \mathcal{M}^{\mathcal{C}}$$

while the counit is

$$c_M : h_{\mathcal{C}}(Y, M\Box_{\mathcal{D}}Y) \to M \quad \text{for} \quad M \in \mathcal{M}^{\mathcal{D}}.$$

The adjointness property means that we have

$$G(c_M)\circ u_{G(M)} = id_{G(M)}, \quad c_{F(N)}\circ F(u_N) = id_{F(N)}; M \in \mathcal{M}^{\mathcal{D}}, N \in \mathcal{M}^{\mathcal{C}} \ (\star).$$

Let Y be a $(B,\mathcal{C})$-quasi-finite object of ${}_B\mathcal{M}^{\mathcal{C}}$ Set $e_{\mathcal{C}}(Y) = h_{\mathcal{C}}(Y,Y)$. By ([3], 23.8), $e_{\mathcal{C}}(Y)$ is a B-coring. Furthermore, Y is an $(e_{\mathcal{C}}(Y),\mathcal{C})$-bicomodule by

$$\eta_Y : Y \to e_{\mathcal{C}}(Y) \otimes_B Y,$$

and there is a ring anti-isomorphism (the dual of the algebra $End^{\mathcal{C}}(Y)$)

$$\Phi_{Y,B} : e_{\mathcal{C}}(Y)^{\star} = Hom_B(e_{\mathcal{C}}(Y), B) \to End^{\mathcal{C}}(Y).$$

By ([3], 38.21), the functor $h_{\mathcal{C}}(Y,-)$ commutes with direct sums and is right exact since it has a right adjoint.

A quasi-Frobenius ring (QF ring) is a left artinian ring with identity for which the left A-module ${}_AA$ is injective ([3], 43.6). Recall that QF rings A are Artinian and injective and cogenerators in the category ${}_A\mathcal{M}$ of left A-modules and in the category $\mathcal{M}_A$ of right A-modules.

3 The main results

Let k be a field, A and B two k-algebras, $\mathcal{C}$ an A-coring which is flat as a left A-module, $\mathcal{D}$ a B-coring which is flat as a left B-module, and Y is a $(\mathcal{D},\mathcal{C})$-bicomodule which is $(B,\mathcal{C})$-quasi-finite. By ([3], 23.7), the functor $h_{\mathcal{C}}(Y,-)$ is left exact if and only if Y is a $(B,\mathcal{C})$ injector as a right $\mathcal{C}$-comodule. So if Y is a $(B,\mathcal{C})$-injector, the functor $h_{\mathcal{C}}(Y,-)$ is exact. By ([3], 23.4), if Y is $(B,\mathcal{C})$-quasi-finite object of ${}_B\mathcal{M}^{\mathcal{C}}$, then Y is flat a left B-module. By ([3], 18.18(2)), $\mathcal{C}$ is $(\mathcal{C},A)$-injective, if $\mathcal{C}$ is considered as a right $\mathcal{C}$-comodule via $\Delta_{\mathcal{C}}$. If A is a QF ring, then by ([3], 21.9), a right comodule is injective if and only if it is coflat.

Assume that B is a QF ring and $\mathcal{D}$ is projective as a left B-module. So $\mathcal{D}$ is injective as a left B-module ([3], 43.6). Let M be a right $\mathcal{D}$-comodule. By ([3], 19.17 (2)), M is a right $\mathcal{D}$-subcomodule of $\mathcal{D}^{(I)}$ for some index set I. From ([3], 18.18 and 18.19), if a right $\mathcal{D}$-comodule M is $(\mathcal{D},B)$-injective, then it is injective in $\mathcal{M}^{\mathcal{D}}$. Since $\mathcal{D}$ is $(\mathcal{D},B)$-injective, $\mathcal{D}$ is injective in $\mathcal{M}^{\mathcal{D}}$. Since $\mathcal{D}$ is a projective left B-module, it is a locally projective left B-module. So by ([3], 19.2), $\mathcal{D}$ satisfies the left α-condition. For the definition of the left α-condition, we refer to ([3], 19.2). Since B is a noetherian ring, from ([3],19.16), we deduce that $\mathcal{M}^{\mathcal{D}}$ is locally noetherian and direct sums of injectives in $\mathcal{M}^{\mathcal{D}}$ are injective. It follows that $\mathcal{D}^{(I)}$ is injective in $\mathcal{M}^{\mathcal{D}}$ for every index set I.

We deduce from all this that if the k-algebra B is a QF ring, and if $\mathcal{D}$ is projective as a left B-module, then a right $\mathcal{D}$-comodule is injective in $\mathcal{M}^{\mathcal{D}}$ if and only if it is a direct summand of $\mathcal{D}^{(I)}$ for some index set I: it is a generalization of the well known fact that if D is a coalgebra over a field, then a right D-comodule is injective in $\mathcal{M}^D$ if and only if it is a direct summand of $D^{(I)}$ for some index set I. A D-comodule of the form $D^{(I)}$ is called a free D-comodule. Division rings are clearly QF rings, and over a division ring, every module is left and right projective.

Definition 3.1 *Let Y be a $(B,\mathcal{C})$-quasi-finite object of ${}_B\mathcal{M}^{\mathcal{C}}$. Set $\mathcal{D} = e_{\mathcal{C}}(Y)$. We say that Y is a $(B,\mathcal{C})$-semi-injector if for any object N in $\mathcal{M}^{\mathcal{D}}$, the functor $h_{\mathcal{C}}(Y,-) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ sends an exact sequence of the form*

$$0 \to N\Box_{\mathcal{D}}Y \to E_1\Box_{\mathcal{D}}Y \to E_2\Box_{\mathcal{D}}Y$$

to an exact sequence.

Lemma 3.2 *Let Y be a $(B,\mathcal{C})$-quasi-finite object of ${}_B\mathcal{M}^{\mathcal{C}}$. Set $\mathcal{D} = e_{\mathcal{C}}(Y)$. Assume that $\mathcal{D}$ is flat as a left B-module. If Y is a $(B,\mathcal{C})$ injector in ${}_B\mathcal{M}^{\mathcal{C}}$, then Y is a $(B,\mathcal{C})$-semi-injector.*

Proof. Consider in $\mathcal{M}^{\mathcal{C}}$ an exact sequence of the form

$$0 \to N\Box_{\mathcal{D}}Y \to E_1\Box_{\mathcal{D}}Y \to E_2\Box_{\mathcal{D}}Y.$$

Thus the sequence

$$0 \to (N\Box_{\mathcal{D}}Y)\Box_{\mathcal{C}}h_{\mathcal{C}}(Y,\mathcal{C}) \to (E_1\Box_{\mathcal{D}}Y)\Box_{\mathcal{C}}h_{\mathcal{C}}(Y,\mathcal{C}) \to (E_2\Box_{\mathcal{D}}Y)\Box_{\mathcal{C}}h_{\mathcal{C}}(Y,\mathcal{C})$$

because the functor $-\Box_{\mathcal{C}}h_{\mathcal{C}}(Y,\mathcal{C})$ is left exact. Since Y is a $(B,\mathcal{C})$ injector in ${}_B\mathcal{M}^{\mathcal{C}}$ and $\mathcal{D}$ is flat as a left B-module, by ([3], 23.7 (1)), the functor $h_{\mathcal{C}}(Y,-)$ is exact, and there is an isomorphism

$$h_{\mathcal{C}}(Y,M) \simeq M\Box_{\mathcal{C}}h_{\mathcal{C}}(Y,\mathcal{C}), \quad \text{for any} \quad M \in \mathcal{M}^{\mathcal{C}}.$$

From the above exact sequence, we get the exact sequence

$$0 \to h_{\mathcal{C}}(Y, N\Box_{\mathcal{D}}Y) \to h_{\mathcal{C}}(Y, E_1\Box_{\mathcal{D}}Y) \to h_{\mathcal{C}}(Y, E_2\Box_{\mathcal{D}}Y).$$

□

Lemma 3.3 *Let B be a k-algebra which is a QF ring. Let Y be a $(B, \mathcal{C})$-quasi-finite object of ${}_B\mathcal{M}^{\mathcal{C}}$ and let $\mathcal{D} = e_{\mathcal{C}}(Y)$. Assume that $\mathcal{D}$ is projective as a left B-module. For every index set I,*

(1) the natural map $\kappa : \mathcal{D}^{(I)} = h_{\mathcal{C}}(Y, Y)^{(I)} \to h_{\mathcal{C}}(Y, Y^{(I)})$ is an isomorphism;

(2) $u_{Y^{(I)}}$ is an isomorphism;

(3) $c_{\mathcal{D}^{(I)}}$ is an isomorphism;

(4) if Y is $(B, \mathcal{C})$-semi-injector in ${}_B\mathcal{M}^{\mathcal{C}}$, then c is a natural isomorphism; in other words, the coinduction functor $G = (-)\square_{\mathcal{D}} Y$ is fully faithful.

Proof. (1) is easy.

(2) It is straightforward to check that the canonical isomorphism $Y^{(I)} \simeq \mathcal{D}^{(I)} \square_{\mathcal{D}} Y$ is nothing else than $(\kappa \square_{\mathcal{D}} id_Y) \circ u_{Y^{(I)}}$. It follows from (1) that $\kappa \square_{\mathcal{D}} id_Y$ is an isomorphism. So $u_{Y^{(I)}}$ is an isomorphism.

(3) Putting $N = Y^{(I)}$ in $(\star)$ and using (1), we find

$$c_{h_{\mathcal{C}}(Y, Y^{(I)})} \circ h_{\mathcal{C}}(Y, u_{Y^{(I)}}) = id_{h_{\mathcal{C}}(Y, Y^{(I)})}, \text{ i.e.,}$$
$$c_{\mathcal{D}^{(I)}} \circ h_{\mathcal{C}}(Y, u_{Y^{(I)}}) = id_{\mathcal{D}^{(I)}}.$$

From (2), $h_{\mathcal{C}}(Y, u_{Y^{(I)}})$ is an isomorphism, hence $c_{\mathcal{D}^{(I)}}$ is an isomorphism.

(4) Take an injective resolution $0 \to N \to E_1 \to E_2$ of a right $\mathcal{D}$-comodule N. Since c is natural, we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & E_1 & \longrightarrow & E_2 \\
 & & \uparrow{\scriptstyle c_N} & & \uparrow{\scriptstyle c_{E_1}} & & \uparrow{\scriptstyle c_{E_2}} \\
0 & \longrightarrow & FG(N) & \longrightarrow & FG(E_1) & \longrightarrow & FG(E_2)
\end{array}$$

The top row is exact. Since the functor $-\square_{\mathcal{D}} Y$ is left exact, we get the sequence of right $\mathcal{C}$-comodules

$$0 \to N \square_{\mathcal{D}} Y \to E_1 \square_{\mathcal{D}} Y \to E_2 \square_{\mathcal{D}} Y.$$

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The bottom row is exact, since Y is a $(B,\mathcal{C})$-semi-injector object and $FG = h_{\mathcal{C}}(Y, -\Box_{\mathcal{D}}Y)$. By the assumptions, E_1 is a direct summand of $\mathcal{D}^{(I)}$ in $\mathcal{M}^{\mathcal{C}}$ for some index set I. By (3), $c_{\mathcal{D}^{(I)}}$ is an isomorphism. We deduce that c_{E_1} is an isomorphism. In the same way, c_{E_2} is an isomorphism. It follows from the five lemma that c_N is an isomorphism. $\Box$

We can now give equivalent conditions for the injectivity of $E \in \mathcal{M}^{\mathcal{D}}$.

Theorem 3.4 *Let B be a k-algebra which is a QF ring. Let Y be a $(B,\mathcal{C})$-quasi-finite object of ${}_B\mathcal{M}^{\mathcal{C}}$. Set $\mathcal{D} = e_{\mathcal{C}}(Y)$. Assume that $\mathcal{D}$ is projective as a left B-module. For $E \in \mathcal{M}^{\mathcal{D}}$, we consider the following statements.*

(1) $E\Box_{\mathcal{D}}Y$ is injective in $\mathcal{M}^{\mathcal{C}}$ and c_E is surjective;

(2) E is injective as a right $\mathcal{D}$-comodule;

(3) $E\Box_{\mathcal{D}}Y$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $Y^{(I)}$, and c_E is bijective;

(4) there exists $Q \in \mathcal{M}^{\mathcal{C}}$ such that Q is a direct summand of some $Y^{(I)}$, and $E \cong h_{\mathcal{C}}(Y,Q)$ in $\mathcal{M}^{\mathcal{D}}$;

(5) $E\Box_{\mathcal{D}}Y$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $Y^{(I)}$.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If Y is a $(B,\mathcal{C})$-semi-injector in ${}_B\mathcal{M}^{\mathcal{C}}$, then (5) $\Rightarrow$ (3); if Y is a $(B,\mathcal{C})$-injector in ${}_B\mathcal{M}^{\mathcal{C}}$, then (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2). Take a monomorphism $f : E \to \mathcal{D}^{(I)}$ in $\mathcal{M}^{\mathcal{D}}$. Then

$$G(f) = f\Box_{\mathcal{D}}id_Y : E\Box_{\mathcal{D}}Y \to \mathcal{D}^{(I)}\Box_{\mathcal{D}}Y \cong Y^{(I)}$$

is also injective, and split in $\mathcal{M}^{\mathcal{C}}$ since $E\Box_{\mathcal{D}}Y$ is injective. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathbf{0} & \longrightarrow & \mathbf{E} & \xrightarrow{f} & \mathcal{D}^{(\mathbf{I})} \\
 & & \uparrow{\scriptstyle c_E} & & \uparrow{\scriptstyle c_{\mathcal{D}^{(I)}}} \\
\mathbf{0} & \longrightarrow & \mathbf{h}_{\mathcal{C}}(\mathbf{Y}, \mathbf{E}\Box_{\mathcal{D}}\mathbf{Y}) & \xrightarrow{FG(f)} & \mathbf{h}_{\mathcal{C}}(\mathbf{Y}, \mathbf{Y}^{(\mathbf{I})})
\end{array}$$

The bottom row is split exact, since any functor, in particular, $h_{\mathcal{C}}(Y,-)$ preserves split exact sequences. By Lemma 3.3(3), $c_{\mathcal{D}^{(I)}}$ is an isomorphism. A diagram chasing tells us that c_E is injective. By assumption, c_E is surjective, so c_E is bijective. We deduce that the top row is isomorphic to the bottom row, and therefore splits. Thus $E \in \mathcal{M}^{\mathcal{D}}$ is injective since $\mathcal{D}^{(I)}$ is an injective object in $\mathcal{M}^{\mathcal{D}}$.

(2) $\Rightarrow$ (3). If E is injective as a right $\mathcal{D}$-comodule, we know that we can find an index set I and $E' \in \mathcal{M}^{\mathcal{D}}$ such that $\mathcal{D}^{(I)} \cong E \oplus E'$. Then obviously

$$Y^{(I)} \cong \mathcal{D}^{(I)} \Box_{\mathcal{D}} Y \cong (E \Box_{\mathcal{D}} Y) \oplus (E' \Box_{\mathcal{D}} Y).$$

Since c is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{(\mathbf{I})} & \xrightarrow{\cong} & \mathbf{E} \oplus \mathbf{E}' \\ c_{\mathcal{D}^{(I)}} \uparrow & & \uparrow c_E \oplus c_{E'} \\ \mathbf{h}_{\mathcal{C}}(\mathbf{Y}, \mathbf{Y}^{(\mathbf{I})}) & \xrightarrow[\cong]{} & \mathbf{h}_{\mathcal{C}}(\mathbf{Y}, \mathbf{E} \Box_{\mathcal{D}} \mathbf{Y}) \oplus \mathbf{h}_{\mathcal{C}}(\mathbf{Y}, \mathbf{E}' \Box_{\mathcal{D}} \mathbf{Y}) \end{array}.$$

From the fact that $c_{\mathcal{D}^{(I)}}$ is an isomorphism, it follows that c_E (and $c_{E'}$) are isomorphisms.

(3) $\Rightarrow$ (4). Take $Q = E \Box_{\mathcal{D}} Y$.

(4) $\Rightarrow$ (2). Let $f : Q \to Y^{(I)}$ be a split monomorphism in $\mathcal{M}^{\mathcal{C}}$. Then

$$h_{\mathcal{C}}(Y, f) : h_{\mathcal{C}}(Y, Q) \cong E \to h_{\mathcal{C}}(Y, Y^{(I)}) \cong \mathcal{D}^{(I)}$$

is also split injective, hence E is injective as a right $\mathcal{D}$-comodule since $\mathcal{D}^{(I)}$ is an injective in $\mathcal{M}^{\mathcal{D}}$.

(4) $\Rightarrow$ (5). If (4) is true, we know from the proof of (4) $\Rightarrow$ (2) that E is a direct summand of some $\mathcal{D}^{(I)}$. So $E \Box_{\mathcal{D}} Y$ is direct summand of $Y^{(I)}$, and we get (5).

Under the assumption that Y is a $(B, \mathcal{C})$-semi-injector in ${}_B\mathcal{M}^{\mathcal{C}}$, (5) $\Rightarrow$ (3) follows from Lemma 2.3(4).

Let us prove (2) $\Rightarrow$ (1) under the assumption that Y is a $(B,\mathcal{C})$-injector in $\mathcal{M}^{\mathcal{C}}$. By the adjointness isomorphism, we have

$$Hom^{\mathcal{C}}(-, E\Box_{\mathcal{D}}Y) = Hom^{\mathcal{D}}(h_{\mathcal{C}}(Y,-), E).$$

Since Y is a $(B,\mathcal{C})$-injector in ${}_B\mathcal{M}^{\mathcal{C}}$, the functor $h_{\mathcal{C}}(Y,-)$ is exact. By (2), E is an injective object of $\mathcal{M}^{\mathcal{D}}$, so the functor $Hom^{\mathcal{D}}(-,E)$ is exact. We deduce that the functor $Hom^{\mathcal{C}}(-,E\Box_{\mathcal{D}}Y)$ is exact. Therefore, $E\Box_{\mathcal{D}}Y$ is an injective object of $\mathcal{M}^{\mathcal{C}}$. Since (2) implies (3), c_E is bijective. □

Note that if B is a division ring, then $\mathcal{D}$ is projective as a left B-module, and Theorem 3.4 can be applied. If $B = A$ is a QF ring, in particular, a division ring, then Theorem 3.4 can be applied in the category ${}_A\mathcal{M}^{\mathcal{C}}$.

We have the following interesting corollaries:

Corollary 3.5 *Let A be a k-algebra and $\mathcal{C}$ an A-coring. Let Y be a $(k,\mathcal{C})$-quasi-finite object of $\mathcal{M}^{\mathcal{C}}$. Set $\mathcal{D} = e_{\mathcal{C}}(Y)$. For $E \in \mathcal{M}^{\mathcal{D}}$, we consider the following statements.*

(1) $E\Box_{\mathcal{D}}Y$ is injective in $\mathcal{M}^{\mathcal{C}}$ and c_E is surjective;

(2) E is injective as a right $\mathcal{D}$-comodule;

(3) $E\Box_{\mathcal{D}}Y$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $Y^{(I)}$, and c_E is bijective;

(4) there exists $Q \in \mathcal{M}^{\mathcal{C}}$ such that Q is a direct summand of some $Y^{(I)}$, and $E \cong h_{\mathcal{C}}(Y,Q)$ in $\mathcal{M}^{\mathcal{D}}$;

(5) $E\Box_{\mathcal{D}}Y$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $Y^{(I)}$.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If Y is a $(k,\mathcal{C})$-semi-injector in $\mathcal{M}^{\mathcal{C}}$, then (5) $\Rightarrow$ (3); if Y is a $(k,\mathcal{C})$-injector in $\mathcal{M}^{\mathcal{C}}$, then (2) $\Rightarrow$ (1).

In ([3]), A (k,C)-quasi-finite object in $\mathcal{M}^C$ is called a quasi-finite object, and a (k,C)-injector in $\mathcal{M}^C$ is called a C-injector. We will call a (k,C)-semi-injector in $\mathcal{M}^C$ a C-semi-injector.

Corollary 3.6 *Let C a coalgebra over k. Let Y be a quasi-finite object of $\mathcal{M}^C$. Set $D = e_C(Y)$. For $X \in \mathcal{M}^D$, we consider the following statements.*

(1) $X\Box_D Y$ is injective in $\mathcal{M}^C$ and c_E is surjective;

(2) X is injective as a right D-comodule;

(3) $X \Box_D Y$ is a direct summand in $\mathcal{M}^C$ of some $Y^{(I)}$, and c_E is bijective;

(4) there exists $Q \in \mathcal{M}^C$ such that Q is a direct summand of some $Y^{(I)}$, and $X \cong h_C(Y, Q)$ in $\mathcal{M}^D$;

(5) $X \Box_D Y$ is a direct summand in $\mathcal{M}^C$ of some $Y^{(I)}$.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If Y is a C-semi-injector in $\mathcal{M}^C$, then (5) $\Rightarrow$ (3); if Y is an injector in $\mathcal{M}^C$, then (2) $\Rightarrow$ (1).

Let H be a Hopf algebra over k and C a coalgebra over k. We say that C is a right H-comodule coalgebra if C is a right H-comodule via $\rho_C : C \to C \otimes H$, $\rho_C(c) = c_{(0)} \otimes c_{(1)}$, and Δ_C and ϵ_C are H-colinear. The last two conditions mean that

$$\Delta(c_{(0)}) \otimes c_{(1)}) = c_{1(0)} \otimes c_{2(0)} \otimes c_{1(1)}c_{2(1)} \quad \text{and} \quad \epsilon_C(c_{(0)})c_{(1)} = \epsilon_C(c)1_H.$$

If C is a right H-comodule coalgebra, we have the smash coproduct coalgebra $C \bowtie H$ which as a vector space is $C \otimes H$, has counit $\epsilon_C \bowtie \epsilon_H$ and comultiplication as follows:

$$\Delta(c \bowtie h) = (c_1 \bowtie c_{2(1)}h_2) \otimes (c_{2(0)} \bowtie h_1).$$

Denote by $\mathcal{M}^{C \bowtie H}$ the category of right $C \bowtie H$-comodules: the morphisms of $\mathcal{M}^{C \bowtie H}$ are the C-colinear and H-colinear maps.

According to [4], a vector space M is a right (C, H)-comodule if it is

- a right C-comodule via $m \mapsto m_{\{0\}} \otimes m_{\{1\}}$,
- a right H–comodule via $m \mapsto m_{[0]} \otimes m_{[1]}$,
- for all $m \in M$ we have

$$m_{[0]\{0\}} \otimes m_{[0]\{1\}} \otimes m_{[1]} = m_{\{0\}[0]} \otimes m_{\{1\}(0)} \otimes m_{\{0\}[1]}m_{\{1\}(1)} \qquad (\star).$$

Denote by $\mathcal{M}^{(C,H)}$ the category of right (C, H) comodules: the morphisms of $\mathcal{M}^{(C,H)}$ are the C-colinear and H-colinear maps. By ([4] Proposition 1.3), we have $\mathcal{M}^{(C,H)} \simeq \mathcal{M}^{C \bowtie H}$. We refer to [9] for the definition of a left (C, H)-comodule.

Corollary 3.7 *Let H be a Hopf algebra over k and C a a right H-comodule coalgebra over k. Let Y be a quasi-finite object of $\mathcal{M}^{C \bowtie H}$. Set $D = e_{C \bowtie H}(Y)$. For $X \in \mathcal{M}^D$, we consider the following statements.*

(1) $X\square_D Y$ *is injective in* $\mathcal{M}^{C\bowtie H}$ *and* c_E *is surjective;*

(2) X *is injective as a right* D*-comodule;*

(3) $X\square_D Y$ *is a direct summand in* $\mathcal{M}^{C\bowtie H}$ *of some* $Y^{(I)}$*, and* c_E *is bijective;*

(4) there exists $Q \in \mathcal{M}^{C\bowtie H}$ *such that* Q *is a direct summand of some* $Y^{(I)}$*, and* $X \cong h_C(Y, Q)$ *in* $\mathcal{M}^D$*;*

(5) $X\square_D Y$ *is a direct summand in* $\mathcal{M}^{C\bowtie H}$ *of some* $Y^{(I)}$*.*

Then (1) $\Rightarrow$ *(2)* $\Leftrightarrow$ *(3)* $\Leftrightarrow$ *(4)* $\Rightarrow$ *(5).*

If Y *is a* $C \bowtie H$*-semi-injector in* $\mathcal{M}^{C\bowtie H}$*, then (5)* $\Rightarrow$ *(3); if* Y *is a* $C \bowtie H$*- injector in* $\mathcal{M}^{C\bowtie H}$*, then (2)* $\Rightarrow$ *(1).*

Let H be a finite dimensional Hopf algebra, C a right H-comodule coalgebra, $C \bowtie H$ the smash coproduct. We know that C is a left $H^\star$-module coalgebra. We denote by $H^{\star+}$ the kernel of the counit of $H^\star$ and we set $(H^{\star+})C = H^{\star+} \rightharpoonup C$, where "$\rightharpoonup$" denotes the left $H^\star$-action on C. Then $(H^{\star+})C$ is a coideal of C and $C/(H^{\star+})C$ is a coalgebra with a trivial left H-module structure. Let $\overline{C} = C/(H^{\star+})C$ be the quotient coalgebra. Then C is a $(\overline{C}, C \bowtie H)$-bicomodule, and as a right $C \bowtie H$-comodule, it is quasi-finite. So the cohom functor $h_{C\bowtie H}(C, -)$ exists. To every object M in $\mathcal{M}^{C\bowtie H}$ we can associate a right $\overline{C}$-comodule $M/(H^{\star+} \rightharpoonup M)$, and we get a functor $\overline{(-)}$ from $\mathcal{M}^{C\bowtie H}$ to $\mathcal{M}^{\overline{C}}$. This functor is a left adjoint to the functor $-\square_{\overline{C}} C : \mathcal{M}^{\overline{C}} \to \mathcal{M}^{C\bowtie H}$. By the uniqueness of adjointness, $h_{C\bowtie H}(C, -)$ is equivalent to $\overline{(-)}$ (see [9] page 5) for the case of a left $C \bowtie H$-comodule). For further informations on smash coproducts, we refer to [6] et [7].

Corollary 3.8 *Let* H *be a finite dimensional Hopf algebra over* k *and* C *a right* H*-comodule coalgebra over* k*. Set* $D = e_{C\bowtie H}(C) = h_{C\bowtie H}(C, C)$*. For* $X \in \mathcal{M}^D$*, we consider the following statements.*

(1) $X\square_D C$ *is injective in* $\mathcal{M}^{C\bowtie H}$ *and* c_E *is surjective;*

(2) X *is injective as a right* D*-comodule;*

(3) $X\square_D Y$ *is a direct summand in* $\mathcal{M}^{C\bowtie H}$ *of some* $C^{(I)}$*, and* c_E *is bijective;*

(4) there exists $Q \in \mathcal{M}^{C\bowtie H}$ *such that* Q *is a direct summand of some* $C^{(I)}$*, and* $X \cong h_{C\bowtie H}(C, Q)$ *in* $\mathcal{M}^D$*;*

(5) $X\square_D C$ *is a direct summand in* $\mathcal{M}^{C\bowtie H}$ *of some* $C^{(I)}$*.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If C is a semi-injector in $\mathcal{M}^{C\bowtie H}$, then (5) $\Rightarrow$ (3); if C is an injector in $\mathcal{M}^{C\bowtie H}$, then (2) $\Rightarrow$ (1).

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ADDITIVITY OF MAPS PRESERVING A SUM OF TRIPLE PRODUCTS ON STANDARD OPERATOR ALGEBRAS

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Abstract

Let $\mathfrak{A}$ be a standard operator algebra on a Banach space of $\dim \mathfrak{X} > 1$ and $\mathfrak{A}'$ an arbitrary ring. In this paper, we prove that if $\phi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective map preserving a sum of triple products of type $abc + acb + bac + cab + bca + cba$ for all $a, b, c \in \mathfrak{A}$, then ϕ is additive.

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1 Standard operator algebra and maps

Let $\mathfrak{X}$ be a Banach space of dimension > 1 and $\mathfrak{B}(\mathfrak{X})$ the algebra of all linear bounded operators on $\mathfrak{X}$. A subalgebra $\mathfrak{A}$ of $\mathfrak{B}(\mathfrak{X})$ is called *standard operator algebra* if it contain all finite rank operators in $\mathfrak{B}(\mathfrak{X})$.

Let $\mathfrak{A}'$ be a ring and $\phi : \mathfrak{A} \to \mathfrak{A}'$ a map. We say that the map ϕ is *additive* if $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in \mathfrak{A}$.

Lu in [?] studied the additivity of maps defined on standard operator algebras preserving sums of double products of type $ab + ba$ for all $a, b \in \mathfrak{A}$. He proved the following theorem.

Theorem 1.1. *Let $\mathfrak{X}$ be a Banach space with* $\dim \mathfrak{X} > 1$, $\mathfrak{A} \subset \mathfrak{B}(\mathfrak{X})$ *a standard operator algebra and $\mathfrak{A}'$ a ring. Suppose $\phi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective map satisfying*

$$\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$$

for all $a, b \in \mathfrak{A}$. Then ϕ is additive.

In this paper we investigate the additivity of maps defined on standard operator algebras preserving sums of triple products of type $abc + acb + bac + cab + bca + cba$, $a, b, c \in \mathfrak{A}$.

2 The main result

Our main result is the following theorem.

Theorem 2.1. *Let $\mathfrak{X}$ be a Banach space with* $\dim \mathfrak{X} > 1$, $\mathfrak{A} \subset \mathfrak{B}(\mathfrak{X})$ *a standard operator algebra and $\mathfrak{A}'$ a ring. Suppose $\phi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective map satisfying*

$$\begin{aligned}\phi(abc + acb + bac + cab + bca + cba) \\ = \phi(a)\phi(b)\phi(c) + \phi(a)\phi(c)\phi(b) + \phi(b)\phi(a)\phi(c) + \phi(c)\phi(a)\phi(b) \\ + \phi(b)\phi(c)\phi(a) + \phi(c)\phi(b)\phi(a)\end{aligned}$$

for all $a, b, c \in \mathfrak{A}$. Then ϕ is additive.

Based on the techniques used by Lu [?] and Martindale [?], let us fix a nontrivial idempotent operator $e_1 \in \mathfrak{A}$ and let $e_2 = 1 - e_1$, where 1 is the identity operator on $\mathfrak{X}$. Then $\mathfrak{A}$ has a Peirce decomposition $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$, where $\mathfrak{A}_{ij} = e_i\mathfrak{A}e_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: (i) $\mathfrak{A}_{ij}\mathfrak{A}_{jl} \subseteq \mathfrak{A}_{il}$ $(i, j, l = 1, 2)$ and (ii) $\mathfrak{A}_{ij}\mathfrak{A}_{kl} = 0$ if $j \neq k$ $(i, j, k, l = 1, 2)$.

We shall organize the proof of Theorem ?? in a series of lemmas. We begin with the following lemma.

Lemma 2.1. *Let $s = s_{11} + s_{12} + s_{21} + s_{22} \in \mathfrak{A}$. Then:*

(i) If $t_{ij}s_{jk} = 0$ for every $t_{ij} \in \mathfrak{A}_{ij}$ $(1 \leq i, j, k \leq 2)$, then $s_{jk} = 0$. Dually, if $s_{ki}t_{ij} = 0$ for every $t_{ij} \in \mathfrak{A}_{ij}$ $(1 \leq i, j, k \leq 2)$, then $s_{ki} = 0$;

(ii) If $s_{ii}a_{ii}b_{ii} + s_{ii}b_{ii}a_{ii} + a_{ii}s_{ii}b_{ii} + b_{ii}s_{ii}a_{ii} + a_{ii}b_{ii}s_{ii} + b_{ii}a_{ii}s_{ii} = 0$ for all $a_{ii}, b_{ii} \in \mathfrak{A}_{ii}$ $(1 \leq i \leq 2)$, then $s_{ii} = 0$;

(iii) If $a_{ij}s_{jj}b_{jj} + a_{ij}b_{jj}s_{jj} = 0$ for every $a_{ij} \in \mathfrak{A}_{ij}$ $(1 \leq i \neq j \leq j)$ and $b_{jj} \in \mathfrak{A}_{jj}$, then $s_{jj} = 0$. Dually, if $a_{ii}s_{ii}b_{ij} + s_{ii}a_{ii}b_{ij} = 0$ for every $a_{ii} \in \mathfrak{A}_{ii}$ and $b_{ij} \in \mathfrak{A}_{ij}$ $(1 \leq i \neq j \leq j)$, then $s_{ii} = 0$.

Proof. (i) The proof of this case can be found in [?, Lemma 2(ii)].
(ii) If $s_{ii}a_{ii}b_{ii}+s_{ii}b_{ii}a_{ii}+a_{ii}s_{ii}b_{ii}+b_{ii}s_{ii}a_{ii}+a_{ii}b_{ii}s_{ii}+b_{ii}a_{ii}s_{ii} = 0$ for all $a_{ii}, b_{ii} \in \mathfrak{A}_{ii}$ $(1 \leq i \leq 2)$, then $s_{ii}(e_iae_i)(e_ibe_i) + s_{ii}(e_ibe_i)(e_iae_i) + (e_iae_i)s_{ii}(e_ibe_i) + (e_ibe_i)s_{ii}(e_iae_i) + (e_iae_i)(e_ibe_i)s_{ii} + (e_ibe_i)(e_iae_i)s_{ii} = 0$ for all $a, b \in \mathfrak{A}$. Since $\mathfrak{A}$ is dense in $\mathfrak{B}(\mathfrak{X})$ under the strong operator topology, let us consider a net $\{b_\alpha\}_{\alpha\in\Lambda} \subset \mathfrak{B}(\mathfrak{X})$ such that $SOT - \lim_\alpha b_\alpha = 1$. The limit in $s_{ii}(e_ib_\alpha e_i)(e_ib_\alpha e_i)+s_{ii}(e_ib_\alpha e_i)(e_ib_\alpha e_i)+(e_ib_\alpha e_i)s_{ii}(e_ib_\alpha e_i)+(e_ib_\alpha e_i)s_{ii}(e_ib_\alpha e_i)+(e_ib_\alpha e_i)(e_ib_\alpha e_i)s_{ii} + (e_ib_\alpha e_i)(e_ib_\alpha e_i)s_{ii} = 0$. This leads to $s_{jj} = 0$.
(iii) This case is proved similarly to the case (ii). □

Lemma 2.2. $\phi(0) = 0$.

Proof. From surjectivity of ϕ there is an element $a \in \mathfrak{A}$ such that $\phi(a) = 0$. This implies that

$$\begin{aligned}
\phi(0) &= \phi(a00 + a00 + 0a0 + 0a0 + 00a + 00a) \\
&= \phi(a)\phi(0)\phi(0) + \phi(a)\phi(0)\phi(0) + \phi(0)\phi(a)\phi(0) \\
&\quad +\phi(0)\phi(a)\phi(0) + \phi(0)\phi(0)\phi(a) + \phi(0)\phi(0)\phi(a) \\
&= 0.
\end{aligned}$$

□

Lemma 2.3. *Let $a, b, c \in \mathfrak{A}$ such that $\phi(c) = \phi(a) + \phi(b)$. Then*

$$\begin{aligned}\phi(cst + cts + sct + tcs + stc + tsc) \\ = \phi(ast + ats + sat + tas + sta + tsa) \\ + \phi(bst + bts + sbt + tbs + stb + tsb)\end{aligned}$$

for all $s, t \in \mathfrak{A}$.

Proof. For arbitrary elements $s, t \in \mathfrak{A}$ we have

$$\begin{aligned} & \phi(cst + cts + sct + tcs + stc + tsc) \\ = \; & \phi(c)\phi(s)\phi(t) + \phi(c)\phi(t)\phi(s) + \phi(s)\phi(c)\phi(t) + \phi(t)\phi(c)\phi(s) \\ & + \phi(s)\phi(t)\phi(c) + \phi(t)\phi(s)\phi(c) \\ = \; & \big(\phi(a) + \phi(b)\big)\phi(s)\phi(t) + \big(\phi(a) + \phi(b)\big)\phi(t)\phi(s) \\ & + \phi(s)\big(\phi(a) + \phi(b)\big)\phi(t) + \phi(t)\big(\phi(a) + \phi(b)\big)\phi(s) \\ & + \phi(s)\phi(t)\big(\phi(a) + \phi(b)\big) + \phi(t)\phi(s)\big(\phi(a) + \phi(b)\big) \\ = \; & \phi(a)\phi(s)\phi(t) + \phi(a)\phi(t)\phi(s) + \phi(s)\phi(a)\phi(t) + \phi(t)\phi(a)\phi(s) \\ & + \phi(s)\phi(t)\phi(a) + \phi(t)\phi(s)\phi(a) + \phi(b)\phi(s)\phi(t) + \phi(b)\phi(t)\phi(s) \\ & + \phi(s)\phi(b)\phi(t) + \phi(t)\phi(b)\phi(s) + \phi(s)\phi(t)\phi(b) + \phi(t)\phi(s)\phi(b) \\ = \; & \phi(ast + ats + sat + tas + sta + tsa) \\ & + \phi(bst + bts + sbt + tbs + stb + tsb). \end{aligned}$$

□

Lemma 2.4. $\phi(a_{ii} + b_{ij}) = \phi(a_{ii}) + \phi(b_{ij})$ *for all $a_{ii} \in \mathfrak{A}_{ii}$ and $b_{ij} \in \mathfrak{A}_{ij}$ $(i \neq j)$.*

Proof. From the surjectivity of ϕ there exists $c \in \mathfrak{A}$ such that $\phi(c) = \phi(a_{ii}) + \phi(b_{ij})$. Hence, for arbitrary elements $s_{ii} \in \mathfrak{A}_{ii}$ and $t_{ij} \in \mathfrak{A}_{ij}$ we have

$$\begin{aligned} & \phi(cs_{ii}t_{ij} + ct_{ij}s_{ii} + s_{ii}ct_{ij} + t_{ij}cs_{ii} + s_{ii}t_{ij}c + t_{ij}s_{ii}c) \\ = \; & \phi(a_{ii}s_{ii}t_{ij} + a_{ii}t_{ij}s_{ii} + s_{ii}a_{ii}t_{ij} + t_{ij}a_{ii}s_{ii} + s_{ii}t_{ij}a_{ii} + t_{ij}s_{ii}a_{ii}) \\ & + \phi(b_{ij}s_{ii}t_{ij} + b_{ij}t_{ij}s_{ii} + s_{ii}b_{ij}t_{ij} + t_{ij}b_{ij}s_{ii} + s_{ii}t_{ij}b_{ij} + t_{ij}s_{ii}b_{ij}) \\ = \; & \phi(a_{ii}s_{ii}t_{ij} + s_{ii}a_{ii}t_{ij}). \end{aligned}$$

From injectivity of ϕ we obtain $cs_{ii}t_{ij} + s_{ii}ct_{ij} + t_{ij}cs_{ii} + s_{ii}t_{ij}c = a_{ii}s_{ii}t_{ij} + s_{ii}a_{ii}t_{ij}$ which implies $c_{ji}s_{ii}t_{ij} = 0$. Thus $c_{ji} = 0$, by Lemma ??(i). Next, for arbitrary elements $s_{ii} \in \mathfrak{A}_{ii}$ and $t_{jj} \in \mathfrak{A}_{jj}$ we have

$$\phi(cs_{ii}t_{jj} + ct_{jj}s_{ii} + s_{ii}ct_{jj} + t_{jj}cs_{ii} + s_{ii}t_{jj}c + t_{jj}s_{ii}c)$$

$$\begin{aligned} &= \phi(a_{ii}s_{ii}t_{jj} + a_{ii}t_{jj}s_{ii} + s_{ii}a_{ii}t_{jj} + t_{jj}a_{ii}s_{ii} + s_{ii}t_{jj}a_{ii} + t_{jj}s_{ii}a_{ii}) \\ &\quad +\phi(b_{ij}s_{ii}t_{jj} + b_{ij}t_{jj}s_{ii} + s_{ii}b_{ij}t_{jj} + t_{jj}b_{ij}s_{ii} + s_{ii}t_{jj}b_{ij} + t_{jj}s_{ii}b_{ij}) \\ &= \phi(s_{ii}b_{ij}t_{jj}). \end{aligned}$$

From the injectivity of ϕ we obtain $s_{ii}ct_{jj} + t_{jj}cs_{ii} = s_{ii}b_{ij}t_{jj}$ which leads to $s_{ii}c_{ij}t_{jj} = s_{ii}b_{ij}t_{jj}$. Therefore $c_{ij} = b_{ij}$, by Lemma ??(i). Also, for arbitrary elements $s_{jj} \in \mathfrak{A}_{jj}$ and $t_{jj} \in \mathfrak{A}_{jj}$ we have

$$\begin{aligned} &\phi(cs_{jj}t_{jj} + ct_{jj}s_{jj} + s_{jj}ct_{jj} + t_{jj}cs_{jj} + s_{jj}t_{jj}c + t_{jj}s_{jj}c) \\ &= \phi(a_{ii}s_{jj}t_{jj} + a_{ii}t_{jj}s_{jj} + s_{jj}a_{ii}t_{jj} + t_{jj}a_{ii}s_{jj} + s_{jj}t_{jj}a_{ii} + t_{jj}s_{jj}a_{ii}) \\ &\quad +\phi(b_{ij}s_{jj}t_{jj} + b_{ij}t_{jj}s_{jj} + s_{jj}b_{ij}t_{jj} + t_{jj}b_{ij}s_{jj} + s_{jj}t_{jj}b_{ij} + t_{jj}s_{jj}b_{ij}) \\ &= \phi(b_{ij}s_{jj}t_{jj} + b_{ij}t_{jj}s_{jj}). \end{aligned}$$

It follows that $cs_{jj}t_{jj} + ct_{jj}s_{jj} + s_{jj}ct_{jj} + t_{jj}cs_{jj} + s_{jj}t_{jj}c + t_{jj}s_{jj}c = b_{ij}s_{jj}t_{jj} + b_{ij}t_{jj}s_{jj}$ which implies $c_{jj}s_{jj}t_{jj} + c_{jj}t_{jj}s_{jj} + s_{jj}c_{jj}t_{jj} + t_{jj}c_{jj}s_{jj} + s_{jj}t_{jj}c_{jj} + t_{jj}s_{jj}c_{jj} = 0$. Thus $c_{jj} = 0$, by Lemma ??(ii). Yet, for arbitrary elements $s_{ij} \in \mathfrak{A}_{ij}$ and $t_{jj} \in \mathfrak{A}_{jj}$ we have

$$\begin{aligned} &\phi(cs_{ij}t_{jj} + ct_{jj}s_{ij} + s_{ij}ct_{jj} + t_{jj}cs_{ij} + s_{ij}t_{jj}c + t_{jj}s_{ij}c) \\ &= \phi(a_{ii}s_{ij}t_{jj} + a_{ii}t_{jj}s_{ij} + s_{ij}a_{ii}t_{jj} + t_{jj}a_{ii}s_{ij} + s_{ij}t_{jj}a_{ii} + t_{jj}s_{ij}a_{ii}) \\ &\quad +\phi(b_{ij}s_{ij}t_{jj} + b_{ij}t_{jj}s_{ij} + s_{ij}b_{ij}t_{jj} + t_{jj}b_{ij}s_{ij} + s_{ij}t_{jj}b_{ij} + t_{jj}s_{ij}b_{ij}) \\ &= \phi(a_{ii}s_{ij}t_{jj}). \end{aligned}$$

This results in $cs_{ij}t_{jj} + s_{ij}ct_{jj} + t_{jj}cs_{ij} + s_{ij}t_{jj}c = a_{ii}s_{ij}t_{jj}$ which leads to $c_{ii}s_{ij}t_{jj} = a_{ii}s_{ij}t_{jj}$. So $c_{ii} = a_{ii}$, by Lemma ??(i). □

Similarly, we prove the following lemma:

Lemma 2.5. *$\phi(a_{ii} + b_{ji}) = \phi(a_{ii}) + \phi(b_{ji})$ for all $a_{ii} \in \mathfrak{A}_{ii}$ and $b_{ji} \in \mathfrak{A}_{ji}$ $(i \neq j)$.*

Lemma 2.6. *$\phi(a_{12}c_{22} + b_{12}d_{22}) = \phi(a_{12}c_{22}) + \phi(b_{12}d_{22})$ for all $a_{12}, b_{12} \in \mathfrak{A}_{12}$ and $c_{22}, d_{22} \in \mathfrak{A}_{22}$.*

Proof. First of all, we note that the following identity is valid

$$\begin{aligned} a_{12}c_{22} + b_{12}d_{22} &= e_1(d_{22} + a_{12})(c_{22} + b_{12}) + e_1(c_{22} + b_{12})(d_{22} + a_{12}) \\ &\quad +(d_{22} + a_{12})e_1(c_{22} + b_{12}) + (c_{22} + b_{12})e_1(d_{22} + a_{12}) \\ &\quad +(d_{22} + a_{12})(c_{22} + b_{12})e_1 + (c_{22} + b_{12})(d_{22} + a_{12})e_1. \end{aligned}$$

Hence, by Lemma ??, we obtain

$$\begin{aligned}
& \phi(a_{12}c_{22}+b_{12}d_{22}) \\
= \; & \phi\big(e_1(d_{22}+a_{12})(c_{22}+b_{12})+e_1(c_{22}+b_{12})(d_{22}+a_{12}) \\
& +(d_{22}+a_{12})e_1(c_{22}+b_{12})+(c_{22}+b_{12})e_1(d_{22}+a_{12}) \\
& +(d_{22}+a_{12})(c_{22}+b_{12})e_1+(c_{22}+b_{12})(d_{22}+a_{12})e_1\big) \\
= \; & \phi(e_1)\phi(d_{22}+a_{12})\phi(c_{22}+b_{12})+\phi(e_1)\phi(c_{22}+b_{12})\phi(d_{22}+a_{12}) \\
& +\phi(d_{22}+a_{12})\phi(e_1)\phi(c_{22}+b_{12})+\phi(c_{22}+b_{12})\phi(e_1)\phi(d_{22}+a_{12}) \\
& +\phi(d_{22}+a_{12})\phi(c_{22}+b_{12})\phi(e_1)+\phi(c_{22}+b_{12})\phi(d_{22}+a_{12})\phi(e_1) \\
= \; & \phi(e_1)\big(\phi(d_{22})+\phi(a_{12})\big)\phi(c_{22}+b_{12}) \\
& +\phi(e_1)\phi(c_{22}+b_{12})\big(\phi(d_{22})+\phi(a_{12})\big) \\
& +\big(\phi(d_{22})+\phi(a_{12})\big)\phi(e_1)\phi(c_{22}+b_{12}) \\
& +\phi(c_{22}+b_{12})\phi(e_1)\big(\phi(d_{22})+\phi(a_{12})\big) \\
& +\big(\phi(d_{22})+\phi(a_{12})\big)\phi(c_{22}+b_{12})\phi(e_1) \\
& +\phi(c_{22}+b_{12})\big(\phi(d_{22})+\phi(a_{12})\big)\phi(e_1) \\
= \; & \phi(e_1)\phi(d_{22})\phi(c_{22}+b_{12})+\phi(e_1)\phi(c_{22}+b_{12})\phi(d_{22}) \\
& +\phi(d_{22})\phi(e_1)\phi(c_{22}+b_{12})+\phi(c_{22}+b_{12})\phi(e_1)\phi(d_{22}) \\
& +\phi(d_{22})\phi(c_{22}+b_{12})\phi(e_1)+\phi(c_{22}+b_{12})\phi(d_{22})\phi(e_1) \\
& +\phi(e_1)\phi(a_{12})\phi(c_{22}+b_{12})+\phi(e_1)\phi(c_{22}+b_{12})\phi(a_{12}) \\
& +\phi(a_{12})\phi(e_1)\phi(c_{22}+b_{12})+\phi(c_{22}+b_{12})\phi(e_1)\phi(a_{12}) \\
& +\phi(a_{12})\phi(c_{22}+b_{12})\phi(e_1)+\phi(c_{22}+b_{12})\phi(a_{12})\phi(e_1) \\
= \; & \phi\big(e_1d_{22}(c_{22}+b_{12})+e_1(c_{22}+b_{12})d_{22}+d_{22}e_1(c_{22}+b_{12}) \\
& +(c_{22}+b_{12})e_1d_{22}+d_{22}(c_{22}+b_{12})e_1+(c_{22}+b_{12})d_{22}e_1\big) \\
& +\phi\big(e_1a_{12}(c_{22}+b_{12})+e_1(c_{22}+b_{12})a_{12}+a_{12}e_1(c_{22}+b_{12}) \\
& +(c_{22}+b_{12})e_1a_{12}+a_{12}(c_{22}+b_{12})e_1+(c_{22}+b_{12})a_{12}e_1\big) \\
= \; & \phi(a_{12}c_{22})+\phi(b_{12}d_{22}).
\end{aligned}$$

□

Lemma 2.7. $\phi(a_{12}+b_{12})=\phi(a_{12})+\phi(b_{12})$ *for all* $a_{12}, b_{12} \in \mathfrak{A}_{12}$.

Proof. By the surjectivity of ϕ, there exists $c \in \mathfrak{A}$ such that $\phi(c)=\phi(a_{12})+\phi(b_{12})$. Hence, for arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{22} \in \mathfrak{A}_{22}$, we have by Lemma ??

$$\phi(cs_{11}t_{22}+ct_{22}s_{11}+s_{11}ct_{22}+t_{22}cs_{11}+s_{11}t_{22}c+t_{22}s_{11}c)$$

$$\begin{aligned} &= \phi(a_{12}s_{11}t_{22} + a_{12}t_{22}s_{11} + s_{11}a_{12}t_{22} + t_{22}a_{12}s_{11} + s_{11}t_{22}a_{12} + t_{22}s_{11}a_{12}) \\ &\quad +\phi(b_{12}s_{11}t_{22} + b_{12}t_{22}s_{11} + s_{11}b_{12}t_{22} + t_{22}b_{12}s_{11} + s_{11}t_{22}b_{12} + t_{22}s_{11}b_{12}) \\ &= \phi(s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}). \end{aligned}$$

It follows that $s_{11}ct_{22} + t_{22}cs_{11} = s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}$ which implies $s_{11}c_{12}t_{22} = s_{11}a_{12}t_{22}+s_{11}b_{12}t_{22}$ and $t_{22}c_{21}s_{11} = 0$. Therefore, $c_{12} = a_{12}+b_{12}$ and $c_{21} = 0$, by Lemma ??(i). Next, for arbitrary elements $s_{22} \in \mathfrak{A}_{22}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\begin{aligned} &\phi(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c) \\ &= \phi(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + s_{22}a_{12}t_{22} + t_{22}a_{12}s_{22} + s_{22}t_{22}a_{12} + t_{22}s_{22}a_{12}) \\ &\quad +\phi(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + s_{22}b_{12}t_{22} + t_{22}b_{12}s_{22} + s_{22}t_{22}b_{12} + t_{22}s_{22}b_{12}) \\ &= \phi(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}), \end{aligned}$$

by Lemma ?? again. We can thus conclude that $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}$ which yields $c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} = 0$. Thus, $c_{22} = 0$, by Lemma ??(ii). Also, for arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned} &\phi(cs_{11}t_{12} + ct_{12}s_{11} + s_{11}ct_{12} + t_{12}cs_{11} + s_{11}t_{12}c + t_{12}s_{11}c) \\ &= \phi(a_{12}s_{11}t_{12} + a_{12}t_{12}s_{11} + s_{11}a_{12}t_{12} + t_{12}a_{12}s_{11} + s_{11}t_{12}a_{12} + t_{12}s_{11}a_{12}) \\ &\quad +\phi(b_{12}s_{11}t_{12} + b_{12}t_{12}s_{11} + s_{11}b_{12}t_{12} + t_{12}b_{12}s_{11} + s_{11}t_{12}b_{12} + t_{12}s_{11}b_{12}) \\ &= 0. \end{aligned}$$

We can then take $cs_{11}t_{12} + s_{11}ct_{12} + t_{12}cs_{11} + s_{11}t_{12}c = 0$ which results in $c_{11}s_{11}t_{12} + s_{11}c_{11}t_{12} = 0$. So, $c_{11} = 0$, by Lemma ??(iii). □

Similarly, we prove the following lemma:

Lemma 2.8. $\phi(a_{21} + b_{21}) = \phi(a_{21}) + \phi(b_{21})$ *for all* $a_{21}, b_{21} \in \mathfrak{A}_{21}$.

Lemma 2.9. $\phi(a_{11} + b_{11}) = \phi(a_{11}) + \phi(b_{11})$ *for all* $a_{11}, b_{11} \in \mathfrak{A}_{11}$.

Proof. By surjectivity of ϕ we may choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in \mathfrak{A}$ such that $\phi(s) = \phi(a_{11}) + \phi(b_{11})$. It follows that, for arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\phi(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c)$$

$$\begin{aligned} &= \phi(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11} + t_{22}s_{11}a_{11}) \\ &\quad +\phi(b_{11}s_{11}t_{22} + b_{11}t_{22}s_{11} + s_{11}b_{11}t_{22} + t_{22}b_{11}s_{11} + s_{11}t_{22}b_{11} + t_{22}s_{11}b_{11}) \\ &= 0. \end{aligned}$$

This shows that $s_{11}ct_{22} + t_{22}cs_{11} = 0$ which implies $s_{11}c_{12}t_{22} = t_{22}c_{21}s_{11} = 0$. Therefore $c_{12} = c_{21} = 0$, by Lemma ??(i). Next, for arbitrary elements $s_{22} \in \mathfrak{A}_{22}$ and $t_{22} \in \mathfrak{A}_{22}$ we have

$$\begin{aligned} &\phi(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c) \\ &= \phi(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11} + t_{22}s_{22}a_{11}) \\ &\quad +\phi(b_{11}s_{22}t_{22} + b_{11}t_{22}s_{22} + s_{22}b_{11}t_{22} + t_{22}b_{11}s_{22} + s_{22}t_{22}b_{11} + t_{22}s_{22}b_{11}) \\ &= 0. \end{aligned}$$

This leads to $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = 0$ which results in $c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} = 0$. Thus, $c_{22} = 0$. Now, for arbitrary elements $s_{12} \in \mathfrak{A}_{12}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\begin{aligned} &\phi(cs_{12}t_{22} + ct_{22}s_{12} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c + t_{22}s_{12}c) \\ &= \phi(a_{11}s_{12}t_{22} + a_{11}t_{22}s_{12} + s_{12}a_{11}t_{22} + t_{22}a_{11}s_{12} + s_{12}t_{22}a_{11} + t_{22}s_{12}a_{11}) \\ &\quad +\phi(b_{11}s_{12}t_{22} + b_{11}t_{22}s_{12} + s_{12}b_{11}t_{22} + t_{22}b_{11}s_{12} + s_{12}t_{22}b_{11} + t_{22}s_{12}b_{11}) \\ &= \phi(a_{11}s_{12}t_{22} + b_{11}s_{12}t_{22}). \end{aligned}$$

This shows that $cs_{12}t_{22} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c = a_{11}s_{12}t_{22} + b_{11}s_{12}t_{22}$ which yields $c_{11}s_{12}t_{22} = a_{11}s_{12}t_{22} + b_{11}s_{12}t_{22}$. So $c_{11} = a_{11} + b_{11}$. □

Similarly, we prove the following lemma:

Lemma 2.10. *$\phi(a_{22} + b_{22}) = \phi(a_{22}) + \phi(b_{22})$ for all $a_{22}, b_{22} \in \mathfrak{A}_{22}$.*

Lemma 2.11. *$\phi(a_{11} + b_{22}) = \phi(a_{11}) + \phi(b_{22})$ for all $a_{11} \in \mathfrak{A}_{11}$ and $b_{22} \in \mathfrak{A}_{22}$.*

Proof. Choose $s = s_{11}+s_{12}+s_{21}+s_{22} \in \mathfrak{A}$ such that $M(s) = M(a_{11})+M(b_{22})$. For arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\begin{aligned} &\phi(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c) \\ &= \phi(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11} + t_{22}s_{11}a_{11}) \\ &\quad +\phi(b_{22}s_{11}t_{22} + b_{22}t_{22}s_{11} + s_{11}b_{22}t_{22} + t_{22}b_{22}s_{11} + s_{11}t_{22}b_{22} + t_{22}s_{11}b_{22}) \\ &= 0. \end{aligned}$$

It follows that $s_{11}ct_{22} + t_{22}cs_{11} = 0$ which implies $s_{11}c_{12}t_{22} = t_{22}c_{21}s_{11} = 0$. So $c_{12} = c_{21} = 0$, by Lemma ??(i). Next, for arbitrary elements $s_{22} \in \mathfrak{A}_{22}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\begin{aligned} & \phi(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c) \\ = \; & \phi(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11} + t_{22}s_{22}a_{11}) \\ & +\phi(b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}) \\ = \; & \phi(b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}). \end{aligned}$$

This shows that $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}$ which yields $(c_{22} - b_{22})s_{22}t_{22} + (c_{22} - b_{22})t_{22}s_{22} + s_{22}(c_{22} - b_{22})t_{22} + t_{22}(c_{22} - b_{22})s_{22} + s_{22}t_{22}(c_{22} - b_{22}) + t_{22}s_{22}(c_{22} - b_{22}) = 0$. So $c_{22} = b_{22}$, by Lemma ??(ii). Now, for arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned} & \phi(cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c) \\ = \; & \phi(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}) \\ & +\phi(b_{22}s_{11}t_{11} + b_{22}t_{11}s_{11} + s_{11}b_{22}t_{11} + t_{11}b_{22}s_{11} + s_{11}t_{11}b_{22} + t_{11}s_{11}b_{22}) \\ = \; & \phi(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}). \end{aligned}$$

It follows that $cs_{11}t_{11}+ct_{11}s_{11}+s_{11}ct_{11}+t_{11}cs_{11}+s_{11}t_{11}c+t_{11}s_{11}c = a_{11}s_{11}t_{11}+ a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}$ which results in $c_{11} = a_{11}$. □

Lemma 2.12. *If $c = c_{11} + c_{12} + c_{21}$ is such that $\phi(c) = \phi(a_{12}) + \phi(b_{21})$, then $c_{11} = 0$, $c_{12} = a_{12}$ and $c_{21} = a_{21}$.*

Proof. In fact, for arbitrary elements $s_{21} \in \mathfrak{A}_{21}$ and $t_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned} & \phi(cs_{21}t_{11} + ct_{11}s_{21} + s_{21}ct_{11} + t_{11}cs_{21} + s_{21}t_{11}c + t_{11}s_{21}c) \\ = \; & \phi(a_{12}s_{21}t_{11} + a_{12}t_{11}s_{21} + s_{21}a_{12}t_{11} + t_{11}a_{12}s_{21} + s_{21}t_{11}a_{12} + t_{11}s_{21}a_{12}) \\ & +\phi(b_{21}s_{21}t_{11} + b_{21}t_{11}s_{21} + s_{21}b_{21}t_{11} + t_{11}b_{21}s_{21} + s_{21}t_{11}b_{21} + t_{11}s_{21}b_{21}) \\ = \; & \phi(a_{12}s_{21}t_{11} + t_{11}a_{12}s_{21} + s_{21}t_{11}a_{12}). \end{aligned}$$

It follows that $cs_{21}t_{11} + s_{21}ct_{11} + t_{11}cs_{21} + s_{21}t_{11}c = a_{12}s_{21}t_{11} + t_{11}a_{12}s_{21} + s_{21}t_{11}a_{12}$ which implies $s_{21}t_{11}c_{12} = s_{21}t_{11}a_{12}$ and $s_{21}c_{11}t_{11} + s_{21}t_{11}c_{11} = 0$. Thus, $c_{11} = 0$ and $c_{12} = a_{12}$, by Lemma ??(i) and (iii), respectively. Now, for arbitrary elements $s_{12} \in \mathfrak{A}_{12}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\phi(cs_{12}t_{22} + ct_{22}s_{12} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c + t_{22}s_{12}c)$$

$$\begin{aligned} &= \phi(a_{12}s_{12}t_{22} + a_{12}t_{22}s_{12} + s_{12}a_{12}t_{22} + t_{22}a_{12}s_{12} + s_{12}t_{22}a_{12} + t_{22}s_{12}a_{12}) \\ &\quad +\phi(b_{21}s_{12}t_{22} + b_{21}t_{22}s_{12} + s_{12}b_{21}t_{22} + t_{22}b_{21}s_{12} + s_{12}t_{22}b_{21} + t_{22}s_{12}b_{21}) \\ &= \phi(b_{21}s_{12}t_{22} + t_{22}b_{21}s_{12} + s_{12}t_{22}b_{21}). \end{aligned}$$

We can then take $cs_{12}t_{22} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c = b_{21}s_{12}t_{22} + t_{22}b_{21}s_{12} + s_{12}t_{22}b_{21}$ which yields $s_{12}t_{22}c_{21} = s_{12}t_{22}b_{21}$. So, $c_{21} = b_{21}$. □

Lemma 2.13. *$\phi(a_{12}+b_{21}) = \phi(a_{12}) + \phi(b_{21})$ for all $a_{12} \in \mathfrak{A}_{12}$ and $b_{21} \in \mathfrak{A}_{21}$.*

Proof. By surjectivity of ϕ we may choose $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathfrak{A}$ such that $\phi(c) = \phi(a_{12}) + \phi(b_{21})$. Hence, for arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned} &\phi(cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c) \\ &= \phi(a_{12}s_{11}t_{11} + a_{12}t_{11}s_{11} + s_{11}a_{12}t_{11} + t_{11}a_{12}s_{11} + s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12}) \\ &\quad +\phi(b_{21}s_{11}t_{11} + b_{21}t_{11}s_{11} + s_{11}b_{21}t_{11} + t_{11}b_{21}s_{11} + s_{11}t_{11}b_{21} + t_{11}s_{11}b_{21}) \\ &= \phi(s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12}) + \phi(b_{21}s_{11}t_{11} + b_{21}t_{11}s_{11}). \end{aligned}$$

Since

$$\begin{aligned} &\phi((c_{11}s_{11}t_{11} + c_{11}t_{11}s_{11} + s_{11}c_{11}t_{11} + t_{11}c_{11}s_{11} + s_{11}t_{11}c_{11} + t_{11}s_{11}c_{11}) \\ &\quad +(s_{11}t_{11}c_{12} + t_{11}s_{11}c_{12}) + (c_{21}s_{11}t_{11} + c_{21}t_{11}s_{11})) \\ &= \phi(s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12}) + \phi(b_{21}s_{11}t_{11} + b_{21}t_{11}s_{11}), \end{aligned}$$

then $c_{11}s_{11}t_{11} + c_{11}t_{11}s_{11} + s_{11}c_{11}t_{11} + t_{11}c_{11}s_{11} + s_{11}t_{11}c_{11} + t_{11}s_{11}c_{11} = 0$, $s_{11}t_{11}c_{12}+t_{11}s_{11}c_{12} = s_{11}t_{11}a_{12}+t_{11}s_{11}a_{12}$ and $c_{21}s_{11}t_{11}+c_{21}t_{11}s_{11} = b_{21}s_{11}t_{11}+b_{21}t_{11}s_{11}$, by Lemma ??. Therefore, $c_{11} = 0$, $c_{12} = a_{12}$ and $c_{21} = b_{21}$, by Lema ??(ii) and (iii), respectively. Next, for arbitrary elements $s_{21} \in \mathfrak{A}_{21}$ and $t_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned} &\phi(cs_{21}t_{11} + ct_{11}s_{21} + s_{21}ct_{11} + t_{11}cs_{21} + s_{21}t_{11}c + t_{11}s_{21}c) \\ &= \phi(a_{12}s_{21}t_{11} + a_{12}t_{11}s_{21} + s_{21}a_{12}t_{11} + t_{11}a_{12}s_{21} + s_{21}t_{11}a_{12} + t_{11}s_{21}a_{12}) \\ &\quad +\phi(b_{21}s_{21}t_{11} + b_{21}t_{11}s_{21} + s_{21}b_{21}t_{11} + t_{11}b_{21}s_{21} + s_{21}t_{11}b_{21} + t_{11}s_{21}b_{21}) \\ &= \phi(a_{12}s_{21}t_{11} + t_{11}a_{12}s_{21} + s_{21}t_{11}a_{12}). \end{aligned}$$

This shows that $cs_{21}t_{11} + s_{21}ct_{11} + t_{11}cs_{21} + s_{21}t_{11}c = a_{12}s_{21}t_{11} + t_{11}a_{12}s_{21} + s_{21}t_{11}a_{12}$ which implies $c_{22}s_{21}t_{11} = 0$. So $c_{22} = 0$. □

Lemma 2.14. *If $d = d_{11}+d_{12}+d_{21}$ is such that $\phi(d) = \phi(a_{11})+\phi(b_{12})+\phi(c_{21})$, then $d_{11} = a_{11}$, $d_{12} = b_{12}$ and $d_{21} = c_{21}$.*

Proof. Write $\phi(d) = \phi(a_{11}+b_{12})+\phi(c_{21})$, by Lemma ??. Then, for arbitrary elements $s_{12} \in \mathfrak{A}_{12}$ and $t_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}
& \phi(ds_{11}t_{22} + dt_{22}s_{11} + s_{11}dt_{22} + t_{22}ds_{11} + s_{11}t_{22}d + t_{22}s_{11}d) \\
= \ & \phi((a_{11} + b_{12})s_{11}t_{22} + (a_{11} + b_{12})t_{22}s_{11} + s_{11}(a_{11} + b_{12})t_{22} \\
& +t_{22}(a_{11} + b_{12})s_{11} + s_{11}t_{22}(a_{11} + b_{12}) + t_{22}s_{11}(a_{11} + b_{12})) \\
& +\phi(c_{21}s_{11}t_{22} + c_{21}t_{22}s_{11} + s_{11}c_{21}t_{22} + t_{22}c_{21}s_{11} + s_{11}t_{22}c_{21} + t_{22}s_{11}c_{21}) \\
= \ & \phi(s_{11}b_{12}t_{22} + t_{22}c_{21}s_{11}).
\end{aligned}$$

We can thus conclude that $s_{11}dt_{22} + t_{22}ds_{11} = s_{11}b_{12}t_{22} + t_{22}c_{21}s_{11}$ which implies $s_{11}d_{12}t_{22} = s_{11}b_{12}t_{22}$ and $t_{22}d_{21}s_{11} = t_{22}c_{21}s_{11}$. Thus, $d_{12} = b_{12}$ and $d_{21} = c_{21}$. Now, for arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{21} \in \mathfrak{A}_{21}$, we have

$$\begin{aligned}
& \phi(ds_{11}t_{21} + dt_{21}s_{11} + s_{11}dt_{21} + t_{21}ds_{11} + s_{11}t_{21}d + t_{21}s_{11}d) \\
= \ & \phi((a_{11} + b_{12})s_{11}t_{21} + (a_{11} + b_{12})t_{21}s_{11} + s_{11}(a_{11} + b_{12})t_{21} \\
& +t_{21}(a_{11} + b_{12})s_{11} + s_{11}t_{21}(a_{11} + b_{12}) + t_{21}s_{11}(a_{11} + b_{12})) \\
& +\phi(c_{21}s_{11}t_{21} + c_{21}t_{21}s_{11} + s_{11}c_{21}t_{21} + t_{21}c_{21}s_{11} + s_{11}t_{21}c_{21} + t_{21}s_{11}c_{21}) \\
= \ & \phi(b_{12}t_{21}s_{11} + s_{11}b_{12}t_{21} + t_{21}a_{11}s_{11} + t_{21}s_{11}a_{11} + t_{21}s_{11}b_{12}).
\end{aligned}$$

We can then get $dt_{21}s_{11} + s_{11}dt_{21} + t_{21}ds_{11} + t_{21}s_{11}d = b_{12}t_{21}s_{11} + s_{11}b_{12}t_{21} + t_{21}a_{11}s_{11}+t_{21}s_{11}a_{11}+t_{21}s_{11}b_{12}$ which results in $t_{21}d_{11}s_{11}+t_{21}s_{11}d_{11} = t_{21}a_{11}s_{11} + t_{21}s_{11}a_{11}$. So, $d_{11} = a_{11}$. □

Lemma 2.15. *$\phi(a_{11} + b_{12} + c_{21}) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21})$ for all $a_{11} \in \mathfrak{A}_{11}$, $b_{12} \in \mathfrak{A}_{12}$ and $c_{21} \in \mathfrak{A}_{21}$.*

Proof. Choose $d = d_{11}+d_{12}+d_{21}+d_{22} \in \mathfrak{A}$ such that $\phi(d) = \phi(a_{11})+\phi(b_{12})+\phi(c_{21})$ and write $\phi(d) = \phi(a_{11} + b_{12}) + \phi(c_{21})$.

For arbitrary elements $s_{11} \in \mathfrak{A}_{11}$ and $t_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned}
& \phi(ds_{11}t_{11} + dt_{11}s_{11} + s_{11}dt_{11} + t_{11}ds_{11} + s_{11}t_{11}d + t_{11}s_{11}d) \\
= \ & \phi((a_{11} + b_{12})s_{11}t_{11} + (a_{11} + b_{12})t_{11}s_{11} + s_{11}(a_{11} + b_{12})t_{11} \\
& +t_{11}(a_{11} + b_{12})s_{11} + s_{11}t_{11}(a_{11} + b_{12}) + t_{11}s_{11}(a_{11} + b_{12})) \\
& +\phi(d_{21}s_{11}t_{11} + d_{21}t_{11}s_{11} + s_{11}d_{21}t_{11} + t_{11}d_{21}s_{11} + s_{11}t_{11}d_{21} + t_{11}s_{11}d_{21}) \\
= \ & \phi(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}
\end{aligned}$$

$$
\begin{aligned}
& +s_{11}t_{11}b_{12}+t_{11}s_{11}b_{12})+\phi(d_{21}s_{11}t_{11}+d_{21}t_{11}s_{11}) \\
= \; & \phi(a_{11}s_{11}t_{11}+a_{11}t_{11}s_{11}+s_{11}a_{11}t_{11}+t_{11}a_{11}s_{11}+s_{11}t_{11}a_{11}+t_{11}s_{11}a_{11}) \\
& +\phi(s_{11}t_{11}b_{12}+t_{11}s_{11}b_{12})+\phi(d_{21}s_{11}t_{11}+d_{21}t_{11}s_{11}).
\end{aligned}
$$

From Lemma ??, we can then get $d_{11}s_{11}t_{11}+d_{11}t_{11}s_{11}+s_{11}d_{11}t_{11}+t_{11}d_{11}s_{11}+s_{11}t_{11}d_{11}+t_{11}s_{11}d_{11} = a_{11}s_{11}t_{11}+a_{11}t_{11}s_{11}+s_{11}a_{11}t_{11}+t_{11}a_{11}s_{11}+s_{11}t_{11}a_{11}+t_{11}s_{11}a_{11}$, $s_{11}t_{11}d_{12}+t_{11}s_{11}d_{12} = s_{11}t_{11}b_{12}+t_{11}s_{11}b_{12}$ and $d_{21}s_{11}t_{11}+d_{21}t_{11}s_{11} = c_{21}s_{11}t_{11}+c_{21}t_{11}s_{11}$. Thus, $d_{11}=a_{11}$, $d_{12}=b_{12}$ and $d_{21}=c_{21}$, by Lemma ??(ii) and (iii), respectively. Next, for arbitrary elements $s_{22}\in\mathfrak{A}_{22}$ and $t_{22}\in\mathfrak{A}_{22}$, we have

$$
\begin{aligned}
& \phi(ds_{22}t_{22}+dt_{22}s_{22}+s_{22}dt_{22}+t_{22}ds_{22}+s_{22}t_{22}d+t_{22}s_{22}d) \\
= \; & \phi\big((a_{11}+b_{12})s_{22}t_{22}+(a_{11}+b_{12})t_{22}s_{22}+s_{22}(a_{11}+b_{12})t_{22} \\
& +t_{22}(a_{11}+b_{12})s_{22}+s_{22}t_{22}(a_{11}+b_{12})+t_{22}s_{22}(a_{11}+b_{12})\big) \\
& +\phi(c_{21}s_{22}t_{22}+c_{21}t_{22}s_{22}+s_{22}c_{21}t_{22}+t_{22}c_{21}s_{22}+s_{22}t_{22}c_{21}+t_{22}s_{22}c_{21}) \\
= \; & \phi(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}+s_{22}t_{22}c_{21}+t_{22}s_{22}c_{21}),
\end{aligned}
$$

by Lemma ??. This show that $ds_{22}t_{22}+dt_{22}s_{22}+s_{22}dt_{22}+t_{22}ds_{22}+s_{22}t_{22}d+t_{22}s_{22}d = b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}+s_{22}t_{22}c_{21}+t_{22}s_{22}c_{21}$ which yields $d_{22}s_{22}t_{22}+d_{22}t_{22}s_{22}+s_{22}d_{22}t_{22}+t_{22}d_{22}s_{22}+s_{22}t_{22}d_{22}+t_{22}s_{22}d_{22}=0$. So $d_{22}=0$. □

Lemma 2.16. *If $f=f_{12}+f_{21}+f_{22}$ is such that $\phi(f)=\phi(b_{12})+\phi(c_{21})+\phi(d_{22})$, then $f_{12}=b_{12}$, $f_{21}=c_{21}$ and $f_{22}=d_{22}$.*

Proof. Write $M(d)=M(b_{12})+M(c_{21}+d_{22})$. Hence, for arbitrary elements $s_{11}\in\mathfrak{A}_{11}$ and $t_{22}\in\mathfrak{A}_{22}$, we have

$$
\begin{aligned}
& \phi(fs_{11}t_{22}+ft_{22}s_{11}+s_{11}ft_{22}+t_{22}fs_{11}+s_{11}t_{22}f+t_{22}s_{11}f) \\
= \; & \phi(b_{12}s_{11}t_{22}+b_{12}t_{22}s_{11}+s_{11}b_{12}t_{22}+t_{22}b_{12}s_{11}+s_{11}t_{22}b_{12}+t_{22}s_{11}b_{12}) \\
& +\phi\big((c_{21}+d_{22})s_{11}t_{22}+(c_{21}+d_{22})t_{22}s_{11}+s_{11}(c_{21}+d_{22})t_{22} \\
& +t_{22}(c_{21}+d_{22})s_{11}+s_{11}t_{22}(c_{21}+d_{22})+t_{22}s_{11}(c_{21}+d_{22})\big) \\
= \; & \phi(s_{11}b_{12}t_{22}+t_{22}c_{21}s_{11}),
\end{aligned}
$$

by Lemma ??. It follows that $s_{11}ft_{22}+t_{22}fs_{11}=s_{11}b_{12}t_{22}+t_{22}c_{21}s_{11}$ which implies $s_{11}f_{12}t_{22}=s_{11}b_{12}t_{22}$ and $t_{22}f_{21}s_{11}=t_{22}c_{21}s_{11}$. Therefore, $f_{12}=b_{12}$ and $f_{21}=c_{21}$. Next, for arbitrary elements $s_{12}\in\mathfrak{A}_{12}$ and $t_{22}\in\mathfrak{A}_{22}$, we have

$$
\phi(fs_{12}t_{22}+ft_{22}s_{12}+s_{12}ft_{22}+t_{22}fs_{12}+s_{12}t_{22}f+t_{22}s_{12}f)
$$

$$\begin{aligned} &= \phi(b_{12}s_{12}t_{22} + b_{12}t_{22}s_{12} + s_{12}b_{12}t_{22} + t_{22}b_{12}s_{12} + s_{12}t_{22}b_{12} + t_{22}s_{12}b_{12}) \\ &\quad +\phi\big((c_{21} + d_{22})s_{12}t_{22} + (c_{21} + d_{22})t_{22}s_{12} + s_{12}(c_{21} + d_{22})t_{22} \\ &\quad +t_{22}(c_{21} + d_{22})s_{12} + s_{12}t_{22}(c_{21} + d_{22}) + t_{22}s_{12}(c_{21} + d_{22})\big) \\ &= \phi(c_{21}s_{12}t_{22} + s_{12}d_{22}t_{22} + t_{22}c_{21}s_{12} + s_{12}t_{22}c_{21} + s_{12}t_{22}d_{22}) \end{aligned}$$

This leads to $fs_{12}t_{22} + s_{12}ft_{22} + t_{22}fs_{12} + s_{12}t_{22}f = c_{21}s_{12}t_{22} + s_{12}d_{22}t_{22} + t_{22}c_{21}s_{12} + s_{12}t_{22}c_{21} + s_{12}t_{22}d_{22}$ which implies $s_{12}f_{22}t_{22} + s_{12}t_{22}f_{22} = s_{12}d_{22}t_{22} + s_{12}t_{22}d_{22}$. So, $f_{22} = d_{22}$, by Lemma ??(iii). □

Lemma 2.17. *$\phi(b_{12} + c_{21} + d_{22}) = \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ for all $b_{12} \in \mathfrak{A}_{12}$, $c_{21} \in \mathfrak{A}_{21}$ and $d_{22} \in \mathfrak{A}_{22}$.*

Proof. Choose $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathfrak{A}$ such that $\phi(f) = \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ and write $\phi(f) = \phi(b_{12}) + \phi(c_{21} + d_{22})$. Hence, for arbitrary elements $s_{12} \in \mathfrak{A}_{12}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\begin{aligned} &\phi(fs_{12}t_{22} + ft_{22}s_{12} + s_{12}ft_{22} + t_{22}fs_{12} + s_{12}t_{22}f + t_{22}s_{12}f) \\ &= \phi(b_{12}s_{12}t_{22} + b_{12}t_{22}s_{12} + s_{12}b_{12}t_{22} + t_{22}b_{12}s_{12} + s_{12}t_{22}b_{12} + t_{22}s_{12}b_{12}) \\ &\quad +\phi\big((c_{21} + d_{22})s_{12}t_{22} + (c_{21} + d_{22})t_{22}s_{12} + s_{12}(c_{21} + d_{22})t_{22} \\ &\quad +t_{22}(c_{21} + d_{22})s_{12} + s_{12}t_{22}(c_{21} + d_{22}) + t_{22}s_{12}(c_{21} + d_{22})\big) \\ &= \phi(c_{21}s_{12}t_{22} + s_{12}d_{22}t_{22} + t_{22}c_{21}s_{12} + s_{12}t_{22}c_{21} + s_{12}t_{22}d_{22}). \end{aligned}$$

Hence, $fs_{12}t_{22} + s_{12}ft_{22} + t_{22}fs_{12} + s_{12}t_{22}f = c_{21}s_{12}t_{22} + s_{12}d_{22}t_{22} + t_{22}c_{21}s_{12} + s_{12}t_{22}c_{21} + s_{12}t_{22}d_{22}$ which implies $s_{12}t_{22}f_{21} = s_{12}t_{22}c_{21}$. Therefore, $f_{21} = c_{21}$. Next, for arbitrary elements $s_{22} \in \mathfrak{A}_{22}$ and $t_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned} &\phi(fs_{22}t_{11} + ft_{11}s_{22} + s_{22}ft_{11} + t_{11}fs_{22} + s_{22}t_{11}f + t_{11}s_{22}f) \\ &= \phi(b_{12}s_{22}t_{11} + b_{12}t_{11}s_{22} + s_{22}b_{12}t_{11} + t_{11}b_{12}s_{22} + s_{22}t_{11}b_{12} + t_{11}s_{22}b_{12}) \\ &\quad +\phi\big((c_{21} + d_{22})s_{22}t_{11} + (c_{21} + d_{22})t_{11}s_{22} + s_{22}(c_{21} + d_{22})t_{11} \\ &\quad +t_{11}(c_{21} + d_{22})s_{22} + s_{22}t_{11}(c_{21} + d_{22}) + t_{11}s_{22}(c_{21} + d_{22})\big) \\ &= \phi(t_{11}b_{12}s_{22} + s_{22}c_{21}t_{11}), \end{aligned}$$

by Lemma ??. We can thus conclude that $s_{22}ft_{11} + t_{11}fs_{22} = t_{11}b_{12}s_{22} + s_{22}c_{21}t_{11}$ which results in $t_{11}f_{12}s_{22} = t_{11}b_{12}s_{22}$. So, $f_{12} = b_{12}$. Also, for arbitrary elements $s_{22} \in \mathfrak{A}_{22}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$\begin{aligned} &\phi(fs_{22}t_{22} + ft_{22}s_{22} + s_{22}ft_{22} + t_{22}fs_{22} + s_{22}t_{22}f + t_{22}s_{22}f) \\ &= \phi(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + s_{22}b_{12}t_{22} + t_{22}b_{12}s_{22} + s_{22}t_{22}b_{12} + t_{22}s_{22}b_{12}) \end{aligned}$$

$$
\begin{aligned}
&+\phi\big((c_{21}+d_{22})s_{22}t_{22}+(c_{21}+d_{22})t_{22}s_{22}+s_{22}(c_{21}+d_{22})t_{22}\\
&+t_{22}(c_{21}+d_{22})s_{22}+s_{22}t_{22}(c_{21}+d_{22})+t_{22}s_{22}(c_{21}+d_{22})\big)\\
=\ &\phi(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22})+\phi(s_{22}t_{22}c_{21}+t_{22}s_{22}c_{21}+d_{22}s_{22}t_{22}+d_{22}t_{22}s_{22}\\
&+s_{22}d_{22}t_{22}+t_{22}d_{22}s_{22}+s_{22}t_{22}d_{22}+t_{22}s_{22}d_{22})\\
=\ &\phi(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22})+\phi(s_{22}t_{22}c_{21}+t_{22}s_{22}c_{21})+\phi(d_{22}s_{22}t_{22}\\
&+d_{22}t_{22}s_{22}+s_{22}d_{22}t_{22}+t_{22}d_{22}s_{22}+s_{22}t_{22}d_{22}+t_{22}s_{22}d_{22}).
\end{aligned}
$$

From Lemma ??, we can get $f_{22}s_{22}t_{22}+f_{22}t_{22}s_{22}+s_{22}f_{22}t_{22}+t_{22}f_{22}s_{22}+s_{22}t_{22}f_{22}+t_{22}s_{22}f_{22}=d_{22}s_{22}t_{22}+d_{22}t_{22}s_{22}+s_{22}d_{22}t_{22}+t_{22}d_{22}s_{22}+s_{22}t_{22}d_{22}+t_{22}s_{22}d_{22}$ which yields $f_{22}=d_{22}$, by Lemma ??(ii). Yet, for arbitrary elements $s_{12}\in\mathfrak{A}_{12}$ and $t_{11}\in\mathfrak{A}_{11}$, we have

$$
\begin{aligned}
&\phi(fs_{12}t_{11}+ft_{11}s_{12}+s_{12}ft_{11}+t_{11}fs_{12}+s_{12}t_{11}f+t_{11}s_{12}f)\\
=\ &\phi(b_{12}s_{12}t_{11}+b_{12}t_{11}s_{12}+s_{12}b_{12}t_{11}+t_{11}b_{12}s_{12}+s_{12}t_{11}b_{12}+t_{11}s_{12}b_{12})\\
&+\phi\big((c_{21}+d_{22})s_{12}t_{11}+(c_{21}+d_{22})t_{11}s_{12}+s_{12}(c_{21}+d_{22})t_{11}\\
&+t_{11}(c_{21}+d_{22})s_{12}+s_{12}t_{11}(c_{21}+d_{22})+t_{11}s_{12}(c_{21}+d_{22})\big)\\
=\ &\phi(c_{21}t_{11}s_{12}+s_{12}c_{21}t_{11}+t_{11}s_{12}c_{21}+t_{11}s_{12}d_{22}).
\end{aligned}
$$

It follows that $ft_{11}s_{12}+s_{12}ft_{11}+t_{11}fs_{12}+t_{11}s_{12}f=c_{21}t_{11}s_{12}+s_{12}c_{21}t_{11}+t_{11}s_{12}c_{21}+t_{11}s_{12}d_{22}$ which implies $f_{11}t_{11}s_{12}+t_{11}f_{11}s_{12}=0$. So $f_{11}=0$, by Lemma ??(iii). □

Lemma 2.18. *$\phi(a_{11}+b_{12}+c_{21}+d_{22})=\phi(a_{11})+\phi(b_{12})+\phi(c_{21})+\phi(d_{22})$ for all $a_{11}\in\mathfrak{A}_{11}$, $b_{12}\in\mathfrak{A}_{12}$, $c_{21}\in\mathfrak{A}_{21}$ and $c_{22}\in\mathfrak{A}_{22}$.*

Proof. Choose $f=f_{11}+f_{12}+f_{21}+f_{22}\in\mathfrak{A}$ such that $\phi(f)=\phi(a_{11})+\phi(b_{12})+\phi(c_{21})+\phi(d_{22})$ and write $\phi(f)=\phi(a_{11}+b_{12})+\phi(c_{21}+d_{22})$. For arbitrary elements $s_{11}\in\mathfrak{A}_{11}$ and $t_{11}\in\mathfrak{A}_{11}$, we have

$$
\begin{aligned}
&\phi(fs_{11}t_{11}+ft_{11}s_{11}+s_{11}ft_{11}+t_{11}fs_{11}+s_{11}t_{11}f+t_{11}s_{11}f)\\
=\ &\phi\big((a_{11}+b_{12})s_{11}t_{11}+(a_{11}+b_{12})t_{11}s_{11}+s_{11}(a_{11}+b_{12})t_{11}\\
&+t_{11}(a_{11}+b_{12})s_{11}+s_{11}t_{11}(a_{11}+b_{12})+t_{11}s_{11}(a_{11}+b_{12})\big)\\
&+\phi\big((c_{21}+d_{22})s_{11}t_{11}+(c_{21}+d_{22})t_{11}s_{11}+s_{11}(c_{21}+d_{22})t_{11}\\
&+t_{11}(c_{21}+d_{22})s_{11}+s_{11}t_{11}(c_{21}+d_{22})+t_{11}s_{11}(c_{21}+d_{22})\big)\\
=\ &\phi(a_{11}s_{11}t_{11}+a_{11}t_{11}s_{11}+s_{11}a_{11}t_{11}+t_{11}a_{11}s_{11}+s_{11}t_{11}a_{11}\\
&+t_{11}s_{11}a_{11}+s_{11}t_{11}b_{12}+t_{11}s_{11}b_{12})+\phi(c_{21}s_{11}t_{11}+c_{21}t_{11}s_{11})
\end{aligned}
$$

$$
\begin{aligned}
= \ & \phi(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} \\
& + t_{11}s_{11}a_{11}) + \phi(s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}) + \phi(c_{21}s_{11}t_{11} + c_{21}t_{11}s_{11}).
\end{aligned}
$$

By Lemma ??, we conclude that $f_{11}s_{11}t_{11}+f_{11}t_{11}s_{11}+s_{11}f_{11}t_{11}+t_{11}f_{11}s_{11}+s_{11}t_{11}f_{11}+t_{11}s_{11}f_{11} = a_{11}s_{11}t_{11}+a_{11}t_{11}s_{11}+s_{11}a_{11}t_{11}+t_{11}a_{11}s_{11}+s_{11}t_{11}a_{11}+t_{11}s_{11}a_{11}$, $s_{11}t_{11}f_{12}+t_{11}s_{11}f_{12} = s_{11}t_{11}b_{12}+t_{11}s_{11}b_{12}$ and $f_{21}s_{11}t_{11}+f_{21}t_{11}s_{11} = c_{21}s_{11}t_{11} + c_{21}t_{11}s_{11}$. Thus, $f_{11} = a_{11}$, $f_{12} = b_{12}$ and $f_{21} = c_{21}$, by Lemma ??(ii) and (iii), respectively. Now, for arbitrary elements $s_{22} \in \mathfrak{A}_{22}$ and $t_{22} \in \mathfrak{A}_{22}$, we have

$$
\begin{aligned}
& \phi(fs_{22}t_{22} + ft_{22}s_{22} + s_{22}ft_{22} + t_{22}fs_{22} + s_{22}t_{22}f + t_{22}s_{22}f) \\
= \ & \phi\big((a_{11} + b_{12})s_{22}t_{22} + (a_{11} + b_{12})t_{22}s_{22} + s_{22}(a_{11} + b_{12})t_{22} \\
& + t_{22}(a_{11} + b_{12})s_{22} + s_{22}t_{22}(a_{11} + b_{12}) + t_{22}s_{22}(a_{11} + b_{12})\big) \\
& + \phi\big((c_{21} + d_{22})s_{22}t_{22} + (c_{21} + d_{22})t_{22}s_{22} + s_{22}(c_{21} + d_{22})t_{22} \\
& + t_{22}(c_{21} + d_{22})s_{22} + s_{22}t_{22}(c_{21} + d_{22}) + t_{22}s_{22}(c_{21} + d_{22})\big) \\
= \ & \phi(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) + \phi(s_{22}t_{22}c_{21} + t_{22}s_{22}c_{21} + d_{22}s_{22}t_{22} \\
& + d_{22}t_{22}s_{22} + s_{22}d_{22}t_{22} + t_{22}d_{22}s_{22} + s_{22}t_{22}d_{22} + t_{22}s_{22}d_{22}) \\
= \ & \phi(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) + \phi(s_{22}t_{22}c_{21} + t_{22}s_{22}c_{21}) + \phi(d_{22}s_{22}t_{22} \\
& + d_{22}t_{22}s_{22} + s_{22}d_{22}t_{22} + t_{22}d_{22}s_{22} + s_{22}t_{22}d_{22} + t_{22}s_{22}d_{22}).
\end{aligned}
$$

By Lemma ??, we can get $f_{22}s_{22}t_{22} + f_{22}t_{22}s_{22} + s_{22}f_{22}t_{22} + t_{22}f_{22}s_{22} + s_{22}t_{22}f_{22} + t_{22}s_{22}f_{22} = d_{22}s_{22}t_{22} + d_{22}t_{22}s_{22} + s_{22}d_{22}t_{22} + t_{22}d_{22}s_{22} + s_{22}t_{22}d_{22} + t_{22}s_{22}d_{22}$. So $f_{22} = d_{22}$. □

Now we are able to prove the Theorem ??. Our proof is similar those presented by Lu [?] and Martindale [?].

Proof of Theorem. Let $a = a_{11}+a_{12}+a_{21}+a_{22}$ and $b = b_{11}+b_{12}+b_{21}+b_{22}$ be arbitrary elements of $\mathfrak{A}$. From lemmas ??, ??, ??, ?? and ??, we compute

$$
\begin{aligned}
& \phi(a + b) \\
= \ & \phi\big((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})\big) \\
= \ & \phi(a_{11} + b_{11}) + \phi(a_{12} + b_{12}) + \phi(a_{21} + b_{21}) + \phi(a_{22} + b_{22}) \\
= \ & \phi(a_{11}) + \phi(b_{11}) + \phi(a_{12}) + \phi(b_{12}) + \phi(a_{21}) + \phi(b_{21}) + \phi(a_{22}) + \phi(b_{22}) \\
= \ & \phi(a_{11}) + \phi(a_{12}) + \phi(a_{21}) + \phi(a_{22}) + \phi(b_{11}) + \phi(b_{12}) + \phi(b_{21}) + \phi(b_{22}) \\
= \ & \phi(a_{11} + a_{12} + a_{21} + a_{22}) + \phi(b_{11} + b_{12} + b_{21} + b_{22})
\end{aligned}
$$

$$= \phi(a) + \phi(b).$$

This show that the map ϕ is additive. The Theorem is proved.

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N*C* - SMARANDACHE CURVE OF INVOLUTE-EVOLUTE CURVE COUPLE ACCORDING TO FRENET FRAME

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Abstract

In this paper, when the unit Darboux vector of the involute curve are taken as the position vectors, the curvature and the torsion of Smarandache curve are calculated. These values are expressed depending upon the evolute curve. Besides, we illustrate example of our main results.

Mathematics Subject Classification (2010): 53A04.
Keywords: evolute curve, involute curve, Smarandache Curves, Frenet invariants

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1 Introduction

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [13]. Special Smarandache curves have been studied by some authors. Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache TB_2 curves in the space E_1^4 [13]. Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case, [1]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves, [5]. Muhammed Çetin, Yılmaz Tuncer and Kemal Karacan investigated special Smarandache curves according to Bishop frame in Euclidean 3-Space and they gave some differential geometric properties of Smarandache curves,[8]. Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in E^3, [3]. Nurten Bayrak, Özcan Bektaş and Salim Yüce studied some special Smarandache curves in E_1^3, [2]. Kemal Taşköprü and Murat Tosun studied special Smarandache curves according to Sabban frame on S^2 [12].

In this paper, special Smarandache curve belonging to α^* involute curve such as N^*C^* drawn by Frenet frame are defined and some related results are given.

2 Preliminaries

The Euclidean 3-space E^3 be inner product given by

$$\langle , \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [9], [10]

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{1}$$

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = \tau T + \kappa B.$$

If we consider the normalization of the Darboux $C = \frac{1}{\|W\|}W$ we have

$$\sin\varphi = \frac{\tau}{\|W\|} \quad \cos\varphi = \frac{\kappa}{\|W\|}$$

and

$$C = \sin\varphi T + \cos\varphi B$$

where $\angle(W, B) = \varphi$.

Definition 2.1 *Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. For $\forall s \in I$, then the curve α^* is called the involute of the curve α, if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^* and $\langle T(s), T^*(s)\rangle = 0$.*

The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ are as follows:

$$\begin{cases} T^* = N \\ N^* = -\cos\varphi T + \sin\varphi B \\ B^* = \sin\varphi T + \cos\varphi B. \end{cases} \tag{2}$$

Theorem 2.1 *The distance between corresponding points of the involute curve in $\mathbb{E}^3$ is*

$$d(\alpha(s), \alpha^*(s)) = |c - s|, c = sbt, \forall s \in I, [9].$$

Theorem 2.2 *Let (α, α^*) be a involute-evolute curves in $\mathbb{E}^3$. For the curvatures and the torsions of the involute-evolute curve (α, α^*) we have,*

$$\begin{cases} \kappa^* = \dfrac{\sqrt{\kappa^2 + \tau^2}}{(c - s)\kappa}, \lambda = c - s \\ \\ \tau^* = \dfrac{\kappa\tau' - \kappa'\tau}{(c - s)\kappa(\kappa^2 + \tau^2)}. \end{cases} \tag{3}$$

Theorem 2.3 *Let (α, α^*) be a involute-evolute curve in $\mathbb{E}^3$. For the vector C^* is the direction of the involute curve α^* we have*

$$C^* = \frac{\varphi'}{\sqrt{\varphi' + \kappa^2 + \tau^2}} N + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\varphi' + \kappa^2 + \tau^2}} C \tag{4}$$

where the vector C is the direction of the Darboux vector W of the evolute curve α, [4].

3 N^*C^*- Smarandache Curve of Involute-Evolute Curve Couple According to Frenet Frame

Let (α, α^*) be a involute-evolute curves in E^3 and $\{T^* N^* B^*\}$ be the Frenet frame of the involute curve α^* at $\alpha^*(s)$. In this case, N^*C^* - Smarandache curve can be defined by

$$\psi(s) = \frac{1}{\sqrt{2}}(N^* + C^*). \tag{5}$$

Solving the above equation by substitution of N^* and C^* from (2) and (4), we obtain

$$\psi(s) = \frac{(-\cos\varphi + a\sin\varphi)T + bN + (\sin\varphi + a\cos\varphi)B}{\sqrt{2}} \tag{6}$$

where

$$a = \frac{\|W\|}{\sqrt{\varphi'^2 + \|W\|^2}}, \quad b = \frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}.$$

The derivative of this equation with respect to s is as follows,

$$\begin{aligned} \psi' = T_\psi \frac{ds_\psi}{ds} &= \frac{1}{\sqrt{2}}[(\varphi' \sin\varphi + a' \sin\varphi + a\varphi' \cos\varphi - \kappa b)T - (\|W\| + b')N \\ &\quad + (\varphi' \cos\varphi + a' \cos\varphi - a\varphi' \sin\varphi + \tau b)B] \end{aligned} \tag{7}$$

and by substitution, we get

$$T_{\psi}(s) = \frac{(\varphi' \sin\varphi + a' \sin\varphi + a\varphi' \cos\varphi - \kappa b)T - (\|W\| + b')N}{\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2}} + \frac{(\varphi' \cos\varphi + a' \cos\varphi - a\varphi' \sin\varphi + \tau b)B}{\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2}} \quad (8)$$

where

$$\frac{ds_{\psi}}{ds} = \sqrt{\frac{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2}{2}}. \quad (9)$$

In order to determine the first curvature and the principal normal of the curve $\psi(s)$, we formalize

$$T'_{\psi}(s) = \frac{\sqrt{2}(\bar{\omega_1} T + \bar{\omega_2} N + \bar{\omega_3} B)}{[(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2]^{\frac{3}{2}}} \quad (10)$$

where

$$\begin{cases} \bar{\omega_1} = (\varphi'' \sin\varphi + \varphi'^2 \cos\varphi + a'' \sin\varphi + 2a'\varphi' \cos\varphi + a\varphi'' \cos\varphi - a\varphi'^2 \sin\varphi - \kappa' b \\ \quad -2\kappa b' + \kappa\|W\|)\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2} - (\varphi' \sin\varphi \\ \quad +a' \sin\varphi + a\varphi' \cos\varphi - \kappa b)\big(\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2}\big)' \\ \bar{\omega_2} = (a\varphi'\|W\| - b\|W\|^2 - \|W\|' + b'')\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2} \\ \quad +(\|W\| + b')\big(\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2}\big)' \\ \bar{\omega_3} = (\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + a'' \cos\varphi - 2a'\varphi' \sin\varphi - a\varphi'' \sin\varphi - a\varphi'^2 \cos\varphi + \tau' b \\ \quad +2\tau b' - \tau\|W\|)\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2} - (\varphi' \cos\varphi \\ \quad +a' \cos\varphi - a\varphi' \sin\varphi + \tau b)\big(\sqrt{(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2}\big)'. \end{cases}$$

The first curvature is

$$\kappa_\psi = \|T'_\psi\| ,$$

$$\kappa_\psi = \frac{\sqrt{2}(\sqrt{\bar{\omega_1}^2 + \bar{\omega_2}^2 + \bar{\omega_3}^2})}{[(\varphi' + a')^2 + (a\varphi' - b\|W\|)^2 + (\|W\| - b')^2]^{\frac{3}{2}}}.$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_\psi = \frac{\bar{\omega_1} T + \bar{\omega_2} N + \bar{\omega_3} B}{\sqrt{\bar{\omega_1}^2 + \bar{\omega_2}^2 + \bar{\omega_3}^2}} , \tag{11}$$

$$B_\psi = \frac{\begin{array}{l}\big[\bar{\omega}_3(-\|W\|+b') - \bar{\omega}_2(\varphi'\cos\varphi + a'\cos\varphi - a\varphi'\sin\varphi + \tau b)\big]T + \big[\bar{\omega}_1(\varphi'\cos\varphi \\ +a'\cos\varphi - a\varphi'\sin\varphi + \tau b) - \bar{\omega}_3(\varphi'\sin\varphi + a'\sin\varphi + a\varphi'\cos\varphi - \kappa b)\big]N \\ +\big[\bar{\omega}_3(\varphi'\sin\varphi + a'\sin\varphi + a\varphi'\cos\varphi - \kappa b) - \bar{\omega}_1(-\|W\|+b')\big]B\end{array}}{\sqrt{(\bar{\omega}_1{}^2+\bar{\omega}_2{}^2+\bar{\omega}_3{}^2)[(\varphi'+a')^2+(a\varphi'-b\|W\|)^2+(\|W\|-b')^2]}} \tag{12}$$

In order to calculate the torsion of the curve ψ, we differentiate

$$\begin{aligned}\psi' = T_\psi \frac{ds_\psi}{ds} &= \frac{1}{\sqrt{2}}\big[(\varphi'\sin\varphi + a'\sin\varphi + a\varphi'\cos\varphi - \kappa b)T - (\|W\|+b')N \\ &\quad +(\varphi'\cos\varphi + a'\cos\varphi - a\varphi'\sin\varphi + \tau b)B\big]\end{aligned}$$

$$\begin{aligned}\psi'' &= \frac{1}{\sqrt{2}}\Big(\big[(\varphi'\sin\varphi + a'\sin\varphi + a\varphi'\cos\varphi - \kappa b)' + \kappa(\|W\|+b')\big]T + \big[\kappa(\varphi'\sin\varphi + a'\sin\varphi \\ &\quad +a\varphi'\cos\varphi - \kappa b) - (\|W\|+b')' + \tau(\varphi'\cos\varphi + a'\cos\varphi - a\varphi'\sin\varphi + \tau b)\big]N \\ &\quad \big[-\tau(\|W\|+b') + (\varphi'\cos\varphi + a'\cos\varphi - a\varphi'\sin\varphi + \tau b)'\big]B\Big).\end{aligned}$$

and thus

$$\psi''' = \frac{(-\bar{\eta}_2\cos\varphi + \bar{\eta}_3\sin\varphi)T + \bar{\eta}_1 N + (\bar{\eta}_2\sin\varphi + \bar{\eta}_3\cos\varphi)B}{\sqrt{2}}$$

where

$$\begin{cases} \bar{\eta_1} = (\varphi'' \sin\varphi + \varphi'^2 \cos\varphi + a'' \sin\varphi + 2a'\varphi' \cos\varphi + a\varphi'' \cos\varphi - a\varphi'^2 \sin\varphi - \kappa' b \\ \qquad -2\kappa b' + \kappa\|W\|)' - \kappa a\varphi'\|W\| + \kappa b\|W\|^2 + \kappa\|W\|' - \kappa b'' \\ \bar{\eta_2} = \varphi'^2\|W\| + 2a'\varphi'\|W\| + a\varphi'\|W\| - b(\kappa\kappa' + \tau\tau') - 2b'\|W\|^2 + \|W\|^3 \\ \bar{\eta_3} = (\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + a'' \cos\varphi - 2a'\varphi' \sin\varphi - a\varphi'' \sin\varphi - a\varphi'^2 \cos\varphi + \tau' b \\ \qquad +2\tau b' - \tau\|W\|)' + \tau a\varphi'\|W\| - \tau b\|W\|^2 - \tau\|W\|' + \tau b'' \end{cases}$$

The torsion is then given by

$$\tau_\psi = \frac{det(\psi', \psi'', \psi''')}{\|\psi' \wedge \psi''\|^2},$$

$$\tau_\psi = \frac{\sqrt{2}(\bar{\eta_1}\bar{\nu_1} + \bar{\eta_2}\bar{\nu_2} + \bar{\eta_3}\bar{\nu_3})}{\sqrt{\bar{\nu_1}^2 + \bar{\nu_2}^2 + \bar{\nu_3}^2}}$$

where

$$
\begin{cases}
\bar{\nu_1} = (\varphi'' \cos\varphi - \varphi'^2 \sin\varphi + a'' \cos\varphi - 2a'\varphi' \sin\varphi - a\varphi'' \sin\varphi - a\varphi'^2 \cos\varphi + \tau' b \\
\qquad +2\tau b' - \tau\|W\|)(-\|W\| + b') - (a\varphi''\|W\| - b\|W\|^2 - \|W\|' + b'')(\varphi' \cos\varphi \\
\qquad +a' \cos\varphi - a\varphi' \sin\varphi + \tau b) \\
\bar{\nu_2} = (\varphi'' \sin\varphi + \varphi'^2 \cos\varphi + a'' \sin\varphi + 2a'\varphi' \cos\varphi + a\varphi'' \cos\varphi - a\varphi'^2 \sin\varphi - \kappa' b \\
\qquad -2\kappa b' + \kappa\|W\|)(\varphi' \cos\varphi + a' \cos\varphi - a\varphi' \sin\varphi + \tau b) - (\varphi'' \cos\varphi - \varphi'^2 \sin\varphi \\
\qquad +a'' \cos\varphi - 2a'\varphi' \sin\varphi - a\varphi'' \sin\varphi - a\varphi'^2 \cos\varphi + \tau' b + 2\tau b' - \tau\|W\|) \\
\qquad (\varphi' \sin\varphi + a' \sin\varphi + a\varphi' \cos\varphi - \kappa b) \\
\bar{\nu_3} = (a\varphi''\|W\| - b\|W\|^2 - \|W\|' + b'')(\varphi' \sin\varphi + a' \sin\varphi + a\varphi' \cos\varphi - \kappa b) \\
\qquad -(\varphi'' \sin\varphi + \varphi'^2 \cos\varphi + a'' \sin\varphi + 2a'\varphi' \cos\varphi + a\varphi'' \cos\varphi - a\varphi'^2 \sin\varphi - \kappa' b \\
\qquad -2\kappa b' + \kappa\|W\|)(-\|W\| + b').
\end{cases}
$$

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3-DIMENSIONAL LEIBNIZ ALGEBRAS

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Abstract

In this paper, we introduce the notion of 2-crossed modules of Leibniz algebras. Furthermore, we discover the relations between various 3-dimensional structures such as 2-crossed modules, crossed squares, cat^2-objects and simplicial objects, in the category of Leibniz algebras.

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Keywords: 2-crossed module, Leibniz algebras, simplicial object.

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Introduction

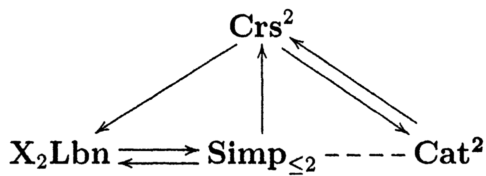
Leibniz algebras are non-associative algebras which generalizes Lie algebras. This close relationship gives rise to the adjoint functors between the category of Lie algebras and of Leibniz algebras: the inclusion functor $i\colon \mathbf{Lie} \to \mathbf{Lbn}$ and conversely, the Liezation functor [8, 10] $Lie_2\colon \mathbf{Lbn} \to \mathbf{Lie}$.

One of the main 2-dimensional structures are crossed modules, which introduced by Whitehead [22] for the case of groups. He defined crossed modules as an algebraic model for homotopy 2-types; with a group homomorphism $\partial\colon G \to G'$ and a group action of G' on G satisfying certain conditions. One step further, as an algebraic models for homotopy 3-types, 2-crossed modules are introduced by Conduché in [13] again for groups. Afterwards, similar notions were defined for different algebraic structures such as (commutative) algebras, Lie algebras, Leibniz algebras, etc. For instance, as we interest in, crossed modules of Leibniz algebras are defined by Casas in [9].

The relation between crossed modules and other 2-dimensional structures in the category of Leibniz algebras can be derived directly from [4, 5, 6] under the modified category of interest perspective. However there is no relation explained between modified category of interest and 2-crossed modules (or other structures we will work on) so far.

The major issue of this paper is to define 2-crossed modules of Leibniz algebras. To get this notion, we follow the similar way to [13] which uses the relation between simplicial groups and (2)-crossed modules. The advantages of this notion are; to have a new algebraic model for (connected) homotopy 3-types and also giving some light to the future studies on 2-crossed modules and other 3-dimensional structures in the category of Leibniz algebras.

We also prove the natural equivalences between the category of 2-crossed modules of Leibniz algebras and some other structures such as crossed squares and cat^2-Leibniz algebras, ending with the diagram:



1 Preliminaries

Throughout this paper, κ will be a fixed commutative ring with the identity.

1.1 Crossed Modules

We follow the notions from [12, 17, 18].

Definition 1 *A Leibniz algebra* $\mathfrak{g}$ *is a* κ*-module equipped with a bilinear map (called Leibniz bracket):*

$$[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that (for all $x, y, z \in \mathfrak{g}$*):*

$$\Big[x, [y, z]\Big] = \Big[[x, y], z\Big] - \Big[[x, z], y\Big].$$

Remark that if $[x, y] = -[y, x]$ then we obtain a Lie algebra structure. Conversely, any Lie algebra is a Leibniz algebra. Therefore the category of Lie algebras is the full subcategory of Leibniz algebras.

Example 2 *Let* A *be an associative algebra. If we define a map* $D\colon A \to A$ *such that (for all* $a, b \in A$*):*

$$D(a\,(Db)) = D(a)\,D(b) = D(D(a)b)$$

then A *forms a Leibniz algebra structure [18] with* $[x, y] = x\,D(y) - D(y)\,x$.

Remark that if $D = id_A$ *then* A *is a Lie algebra. Moreover if* D *is idempotent or if it satisfies* $D(ab) = D(a)b + aD(b)$ *and* $D^2(a) = 0$ *for all* $a, b \in A$*, then* A *forms a Leibniz algebra structure again, with the operation given above.*

Definition 3 *If there exists a Leibniz action of* $\mathfrak{h}$ *on* $\mathfrak{g}$ *then we have semi-direct product Leibniz algebra* $\mathfrak{g} \rtimes \mathfrak{h}$ *with the form* $(l, l' \in \mathfrak{h}$ *and* $r, r' \in \mathfrak{g})$*:*

$$(l,r) + (l',r') = (l+l', r+r')$$
$$\big[(l,r),(l',r')\big] = \big[[l,l'],[r,r'] + {}^{l}r' - r^{l'}\big]$$

Definition 4 *A crossed module of Leibniz algebras [11] is given by a Leibniz algebra homomorphism* $\partial\colon L \to R$ *together with the Leibniz action of* R *on* L *such that the following Peiffer relations hold:*

XM1) $\partial({}^{r}l) = [r, \partial(l)]$ *and* $\partial(l^{r}) = [\partial(l), r]$

XM2) ${}^{\partial(r)}r' = r^{\,\partial(r')} = [r, r']$

for all $r, r' \in R$ *and* $l \in L$.

Therefore we have the category of crossed modules of Leibniz algebras, denoted by **XLbn**. Morphisms and compositions of this category can be defined in a similar sense of Lie algebras.

Example 5 *Let* $\mathfrak{g}$ *be a Leibniz algebra and* $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$. *By using the conjugate action:*

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{h} & \to & \mathfrak{h} \\ (g,h) & \mapsto & [g,h] \end{array} \qquad \begin{array}{ccc} \mathfrak{h} \times \mathfrak{g} & \to & \mathfrak{h} \\ (h,g) & \mapsto & [h,g] \end{array}$$

the inclusion map $i\colon \mathfrak{h} \to \mathfrak{g}$ *defines a crossed module.*

Example 6 *Let* L_0 *be a Leibniz algebra and* $\psi : L_1 \to L_1'$ *be an* L_0*-module homomorphism There exists an action of* $L_1' \rtimes L_0$ *on* L_1 *defined by:*

$$\begin{array}{ccc} (L_1' \rtimes L_0) \times L_1 & \to & L_1 \\ ((l_1', l_0), l_1) & \mapsto & [l_0, l_1] \end{array} \qquad \begin{array}{ccc} L_1 \times (L_1' \rtimes L_0) & \to & L_1 \\ (l_1, (l_1', l_0)) & \mapsto & [l_1, l_0] \end{array}$$

Then:

$$\begin{array}{cccc} \partial\colon & L_1 & \to & L_1' \rtimes L_0 \\ & l_1 & \mapsto & (\psi(l_1), 0) \end{array}$$

is a crossed module of Leibniz algebras.

1.2 Simplicial Leibniz Algebras

We recall some simplicial data from [19].

Definition 7 *A simplicial Leibniz algebra $\mathfrak{L}$ is a collection of Leibniz algebras $\{L_n\colon n \in \mathbb{N}\}$ together with Leibniz algebra morphisms (called faces and degenarices):*

$$\begin{array}{lcll} d_i^{n-1} & : & L_n \longrightarrow L_{n-1} & , \quad 0 \leq i \leq n-1 \\ s_j^n & : & L_n \longrightarrow L_{n+1} & , \quad 0 \leq j \leq n \end{array}$$

These homomorphisms are to satisfy the simplicial identities:

$$\begin{array}{lll} (i) & d_i d_j = d_{j-1} d_i & \text{if } i < j \\ (ii) & s_i s_j = s_{j+1} s_i & \text{if } i \leq j \\ (iii) & d_i s_j = s_{j-1} d_i & \text{if } i < j \\ & d_j s_j = d_{j+1} s_j = id & \\ & d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \end{array} \tag{1}$$

We have thus defined the category of simplicial Leibniz algebras **Simp**.

Definition 8 *The category of k-truncated simplicial Leibniz algebras is the full subcategory of* **Simp** *defined with the finite number of Leibniz algebras L_i $(i \leq k)$, denoted by* $\mathbf{Tr_k Simp}$.

Definition 9 *For a simplicial Leibniz algebra $\mathfrak{L}$, the Moore complex (NL, ∂) is the chain complex defined by:*

$$NL_n = \bigcap_{i=0}^{n-1} Ker(d_i^n)$$

with the morphisms $\partial_n\colon NL_n \to NL_{n-1}$ induced from d_n^{n-1} by restriction.

We call a Moore Complex with length n, iff NL_i is equal to $\{0\}$, for each $i > n$. We denote the category of simplicial objects with Moore Complex of length n by $\boldsymbol{Simp}(\mathcal{C})_{\leq n}$.

This leads us to define the (co)skeleton functors [7]:

$$\mathbf{Tr}_k\mathbf{Simp} \underset{\mathrm{cost}_k}{\longrightarrow} \mathbf{Simp} \underset{st_k}{\longleftarrow} \mathbf{Tr}_k\mathbf{Simp} \tag{2}$$

Lemma 10 *The categories* **XLbn** *and* $\mathbf{Simp}_{\leq 1}$ *are naturally equivalent.*

Proof. Since the category of crossed modules of Leibniz algebras is modified category of interest [6], this lemma is just a corollary of [5]. ■

1.3 Cat^1-Leibniz Algebras

This notion is defined first in [16] for the case of groups.

Definition 11 *Let $s, t : \mathfrak{g} \to \mathfrak{g}$ be Leibniz algebra homomorphisms. We call the triple $(\mathfrak{g}, s, t)$ a cat^1-Leibniz algebra [11] if it satisfies:*

$$\begin{array}{ll} i) & st = t \text{ and } ts = s \\ ii) & [\ker s, \ker t] = 0 = [\ker t, \ker s] \end{array}$$

Therefore we have the category of cat^1-Leibniz algebras, **Cat^1**.

Theorem 12 *The categories* **Cat^1** *and* **XLbn** *are naturally equivalent.*

Proof. Clear from [4], in the sense of modified category of interest aspect. Also: Find detailed calculations in [21]. ■

2 3-Dimensional Leibniz Algebras

In this section we examine some 3-dimensional structures in **Lbn**.

2.1 2-Crossed Modules

Definition 13 *A 2-crossed module $(L, M, N, \partial_1, \partial_2)$ of Leibniz algebras, is defined by a complex of Leibniz algebras $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$, together with actions of N on M and L, such that ∂_1 and ∂_2 are module maps, where N acts on itself by conjugation. We also have two N-bilinear functions (the Peiffer liftings):*

$$\{\,,\,\}_1 : M \times M \longrightarrow L \qquad \{\,,\,\}_2 : M \times M \longrightarrow L$$

satisfying the axioms, for all $n \in N$, $m, m_0, m_1 \in M$ and $l, l_0, l_1 \in L$:

1. $\partial_2 \{m_0, m_1\}_1 = (m_0)^{\partial_1(m_1)} - [m_0, m_1]$
2. $\partial_2 \{m_0, m_1\}_2 = {}^{\partial_1(m_0)}(m_1) - [m_0, m_1]$

3. $\{\partial_2(l_0), \partial_2(l_1)\}_i = -[l_0, l_1]$, $i = 1, 2$

4. $\{\partial_2(l), m\}_1 = l^{\partial_1(m)} - l^m$,
$\{\partial_2(l), m\}_2 = -l^m$

5. $\{m, \partial_2(l)\}_1 = -\,{}^m l$,
$\{m, \partial_2(l)\}_2 = {}^{\partial_1(m)}l - {}^m l$

6. $\{m_1, m_2\}_i^n = \{m_1^n, m_2\}_i - \{m_1, {}^n m_2\}_i$, $i = 1, 2$

7. ${}^n\{m_1, m_2\}_1 = \{{}^n m_1, m_2\}_1 - \{{}^n m_2, m_1\}_2$,
${}^n\{m_1, m_2\}_2 = \{{}^n m_1, m_2\}_2 - \{{}^n m_2, m_1\}_1$

8. $$\{m_0, [m_1, m_2]\}_1 = \{[m_0, m_1], m_2\}_1 - \{[m_0, m_2], m_1\}_1 + \{m_0, m_1\}_1^{\partial_1(m_2)} - \{m_0, m_2\}_1^{\partial_1(m_1)}$$

9. $$\{[m_0, m_1], m_2\}_1 = \{m_0, [m_1, m_2]\}_1 + \{[m_0, m_2], m_1\}_1 - \{m_0, m_1\}_1^{\partial_1(m_2)} + \{m_0, m_2\}_1^{\partial_1(m_1)}$$

10. $$\{m_0, [m_1, m_2]\}_2 = \{[m_0, m_1], m_2\}_2 - \{[m_0, m_2], m_1\}_1 - \{m_0, m_2\}_2^{\partial_1(m_1)} - {}^{\partial_1(m_0)}\{m_1, m_2\}_2$$

11. $$\{[m_0, m_1], m_2\}_2 = \{m_0, [m_1, m_2]\}_2 + \{[m_0, m_2], m_1\}_1 + \{m_0, m_2\}_2^{\partial_1(m_1)} + {}^{\partial_1(m_0)}\{m_1, m_2\}_2$$

We denote the category of 2-crossed modules by $\mathbf{X_2Lbn}$.

Remark 14 *∂_2 is a crossed module, however ∂_1 is not; see* (3).

Lemma 15 *Let $\mathcal{L}$ be a simplicial Leibniz algebra with Moore complex of length* 2. *Define:*

$$L = NL_2, \quad M = NL_1, \quad N = NL_0 = L_0.$$

If we define the actions N on M with:

$$[s_0(n), m] \quad [m, s_0(n)]$$

and N on L with

$$[s_1 s_0 (n), l] \quad [m, s_1 s_0 (l)]$$

with the Peiffer liftings $\{\ ,\ \}_{1,2} : M \times M \to L$, *such that:*

$$\{m_0, m_1\}_1 = [s_1 (m_0), s_0 (m_1) - s_1 (m_1)]$$
$$\{m_0, m_1\}_2 = [s_0 (m_0) - s_1 (m_0), s_1 (m_1)]$$

then $(L, M, N, \partial_1, \partial_2)$ becomes a 2-crossed module where ∂_i is defined as the restriction of d_i.

Proof. By defining hyper crossed complex pairings (see [7] for groups) we get the following relations ($m, m_0, m_1 \in M$, $l, l_0, l_1 \in L$):

$$[l_0, s_0 d_2 (l_1)] = 0 = [s_0 d_2 (l_0), l_1]$$
$$[l_0, s_1 d_2 (l_1) - l_1] = 0 = [s_1 d_2 (l_0) - l_0, l_1]$$
$$[s_1 s_0 d_1 (m) - s_0 (m), l] = 0 = [l, s_1 s_0 d_1 (m) - s_0 (m)]$$
$$[s_0 (m) - s_1 (m), s_1 d_2 (l) - l] = 0 = [s_1 d_2 (l) - l, s_0 (m) - s_1 (m)]$$
$$[s_1 (m), s_0 d_2 (l) - s_1 d_2 (l) + l] = 0 = [s_0 d_2 (l) - s_1 d_2 (l) + l, s_1 (m)]$$
$$[s_1 d_2 (l_0), s_0 d_2 (l_1) - s_1 d_2 (l_1)] + [l_0, l_1] = 0 = [s_0 d_2 (l_0) - s_1 d_2 (l_0), s_1 d_2 (l_1)] + [l_0, l_1]$$

Therefore the conditions given in Definition 13 are satisfies. We will not give the clear calculations since the similar ones are already done in [1, 3, 20] for groups, commutative algebras and Lie algebras. ■

As a generalization of the previous lemma we can give the following:

Lemma 16 *Let $\mathcal{L}$ be any simplicial Leibniz algebra. Then the complex:*

$$NL_2/\partial_3(NL_3 \cap D_3) \xrightarrow{\overline{\partial}_2} NL_1 \xrightarrow{\partial_1} NL_0$$

defines a 2-crossed module with Peiffer liftings:

$$\{m_0, m_1\}_1 = \overline{[s_1 (m_0), s_0 (m_1) - s_1 (m_1)]}$$
$$\{m_0, m_1\}_2 = \overline{[s_0 (m_0) - s_1 (m_0), s_1 (m_1)]}$$

where the overlines denote the cosets.

Theorem 17 *The categories* $\mathbf{X_2Lbn}$ *and* $\mathbf{Simp}_{\leq 2}$ *are naturally equivalent.*

Proof. From Lemma 15 we have the functor: $X_2 \colon \mathbf{Simp}_{\leq 2} \to \mathbf{X_2Lbn}$. Conversely let:

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

be a 2-crossed module of Leibniz algebras. If we define the action of M on L with being:

$$\begin{aligned} {}^{m}l &= {}^{\partial_1(m)}\, l - \{m, \partial_2(l)\}_2 \\ l^{m} &= l^{\partial_1(m)} - \{\partial_2(l), m\}_1 \end{aligned} \tag{3}$$

we get the semi-direct product Leibniz algebras $L \rtimes M$ and $M \rtimes N$. With a similar idea to [14] we can define face and degeneracy morphisms satisfying the simplicial identities (1) which lead us to get a 2-truncated simplicial Leibniz algebra. Finally, by using the (co)skeleton functors (2) we get a simplicial Leibniz algebra with Moore complex of length 2; therefore we get the functor $S_2 \colon \mathbf{X_2Lbn} \to \mathbf{Simp}_{\leq 2}$. ∎

2.2 Crossed Squares

Definition 18 *A crossed square of Leibniz algebras:*

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \delta} \\ M & \xrightarrow[\delta']{} & P \end{array}$$

consists of four Leibniz algebra homomorphisms such that the diagram above commutes, moreover with the actions of P on M, N, L (therefore N acts on L via δ; similarly M on L) and also with κ-bilinear functions h, h° : $M \times N \to L$ such that satisfying:

1. *The morphisms $\lambda, \lambda', \delta, \delta', \delta'\lambda$ and $\delta\lambda'$ are crossed modules,*
   The morphisms λ and λ' preserve the actions of P on L,

2. $\lambda h^{\circ}(x,y) = {}^{y}x$, $\lambda' h^{\circ}(x,y) = y^{x}$
   $\lambda h(x,y) = x^{y}$, $\lambda' h(x,y) = {}^{x}y$,

3. $h^{\circ}(\lambda(l), y) = {}^{y}l$, $h^{\circ}(x, \lambda'(l)) = l^{x}$
   $h(\lambda(l), y) = l^{y}$, $h(x, \lambda'(l)) = {}^{x}l$,

4. $h([m,m'], n) = (h(m,n))^{m'} + {}^{m}(h(m',n))$
   $h^{\circ}([m,m'], n) = (h^{\circ}(m,n))^{m'} - (h^{\circ}(m',n))^{m}$,

5. $h(m,[n,n']) = (h(m,n'))^{n} + (h(m,n))^{n'}$
   $h^{\circ}(m,[n,n']) = {}^{n}(h^{\circ}(m,n')) + (h^{\circ}(m,n))^{n'}$,

6. ${}^{p}(h^{\circ}(m,n)) = h^{\circ}(m, {}^{p}n) - h({}^{p}m, n)$
   ${}^{p}(h(m,n)) = h({}^{p}m, n) - h^{\circ}(m, {}^{p}n)$,

7. $(h^{\circ}(m,n))^{p} = h^{\circ}(m, n^{p}) - h({}^{p}m, n)$
   $(h(m,n))^{p} = h(m^{p}, n) - h(m, {}^{p}n)$,

for all $l \in L$, $m, m' \in M$, $n, n' \in N$, $p \in P$ *and* $k \in \kappa$.

We denote the category of crossed squares by $\mathbf{Crs}^2$.

Example 19 *Let P be a Leibniz algebra and M, N be any two ideals of P. If we define:*

$$h\colon (m,n) \in M \times N \mapsto [m,n] \in L$$
$$h^{\circ}\colon (m,n) \in M \times N \mapsto [n,m] \in L$$

then:

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\delta} \\ M & \xrightarrow[\delta']{} & P \end{array}$$

is a crossed square where $L = M \cap N$ and $\lambda, \lambda', \delta, \delta'$ are the inclusion maps.

Example 20 *Let $M \xrightarrow{\delta'} P$ and $N \xrightarrow{\delta} P$ be two crossed modules of Leibniz algebras. Consider the non-abelian tensor product $M \otimes N$. Here M and N acts on each other via P [15]. Define the morphisms:*

$$h\colon \begin{array}{ccc} M \times N & \to & M \otimes N \\ (m,n) & \mapsto & m \otimes n \end{array} \qquad h^{\circ}\colon \begin{array}{ccc} M \times N & \to & M \otimes N \\ (m,n) & \mapsto & n \otimes m \end{array}$$

There exists an action of P on $M \otimes N$ with:

$$\begin{array}{ll} {}^{p}(m \otimes n) = ({}^{p}m \otimes^{p} n), & {}^{p}(n \otimes m) = ({}^{p}n \otimes^{p} m), \\ (m \otimes n)^{p} = (m^{p} \otimes n^{p}), & (n \otimes m)^{p} = (n^{p} \otimes m^{p}) \end{array}$$

for all $m \otimes n,\ n \otimes m \in M \otimes N$, $p \in P$.

Then we have a crossed square:

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\delta} \\ M & \xrightarrow[\delta']{} & P \end{array}$$

where $\lambda(m\otimes n) = m^{n}$, $\lambda(n\otimes m) = {}^{n}m$, $\lambda'(m\otimes n) = {}^{m}n$ and $\lambda'(n\otimes m) = m^{n}$, for all $m \otimes n,\ n \otimes m \in M \otimes N$.

2.3 Cat^2-Leibniz Algebras

Definition 21 *A cat^2-Leibniz algebra (L, s_1, t_1, s_2, t_2) consists of cat^1-Leibniz algebras (L, s_i, t_i) such that satisfying the relations: $s_i s_j = s_j s_i$, $t_i t_j = t_j t_i$, $s_i t_j = t_j s_i$, $i, j = 1, 2$ for $i \neq j$.*

We denote the category of cat^2-Leibniz algebras by **Cat**^{2}.

Theorem 22 *The categories* **Cat**^{2} *and* **Crs**^{2} *are naturally equivalent.*

Proof. Let (L, s_1, t_1, s_2, t_2) be a *cat*^{2}-Leibniz algebra. By using the inclusion maps, we have the crossed square:

$$\begin{array}{ccc} Kers_1 \bigcap Kers_2 & \longrightarrow & Kers_2 \\ \Big\downarrow & & \Big\downarrow \\ Kers_1 & \longrightarrow & P \end{array}$$

therefore we have the functor $Sq_2\colon \mathbf{Cat^2} \to \mathbf{Crs^2}$. Conversely, let:

$$\begin{array}{ccc} L & \xrightarrow{\ \lambda'\ } & N \\ {\scriptstyle\lambda}\Big\downarrow & & \Big\downarrow{\scriptstyle\delta} \\ M & \xrightarrow[\ \delta'\ ]{} & P \end{array}$$

be a crossed square. We can construct cat^1 Leibniz algebras $(L \rtimes M, s_1, t_1)$ and $(N \rtimes P, s_2, t_2)$ by using crossed modules $L \xrightarrow{\lambda} M$ and $N \xrightarrow{\delta} P$. Also there exist an action of $N \rtimes P$ on $L \rtimes M$ with:

$$\begin{array}{rcl} (N \rtimes P) \times (L \rtimes M) & \longrightarrow & L \rtimes M \\ ((n,p),(l,m)) & \longmapsto & ({}^{n}l + {}^{p}l + h^{\circ}(m,n),\ {}^{p}m) \end{array}$$

and

$$\begin{array}{rcl} (N \rtimes P) \times (L \rtimes M) & \longrightarrow & L \rtimes M \\ ((n,p),(l,m)) & \longmapsto & (l^{n} + l^{p} + h(m,n), m^{p}) \end{array}$$

Thus $(L \rtimes M) \rtimes (N \rtimes P)$ is a Leibniz algebra and finally

$$\Big((L \rtimes M) \rtimes (N \rtimes P), s_1, t_1, s_2, t_2\Big)$$

is a cat^2-Leibniz algebra, obtained from cat^1-Leibniz algebras $(L \rtimes M, s_1, t_1)$ and $(N \rtimes P, s_2, t_2)$. So we get the functor $C_2\colon \mathbf{Crs^2} \to \mathbf{Cat^2}$. ■

Lemma 23 *Let*

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\delta} \\ M & \xrightarrow[\delta']{} & P \end{array}$$

be a crossed square. Then the complex

$$L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P$$

defines a 2-crossed module with Peiffer liftings:

$$\{(m,n),(m',n')\}_1 = h^{\circ}(m',n)$$
$$\{(m,n),(m',n')\}_2 = h(m,n')$$

for all $(m,n),(m',n') \in M \rtimes N$*; where:*

$$\partial_2(l) = (-\lambda(l), \lambda'(l))$$
$$\partial_1(m'',n'') = \delta'(m'') + \delta(n'')$$

for all $l \in L$, $(m'',n'') \in M \rtimes N$. *Here* $M \rtimes N$ *acts on* L *with:*

$${}^{(m,n)}l = {}^{n}l, \qquad l^{(m,n)} = l^{n}$$

for all $(m,n) \in M \rtimes N$ *and* $l \in L$.

Therefore we have the functor $(-)_2\colon \mathbf{Crs^2} \to \mathbf{X_2Mod}$.

Proof. Since

$$\begin{aligned} \partial_2\{(m,n),(m',n')\}_1 &= (-\lambda h^{\circ}(m',n), \lambda' h^{\circ}(m',n)) \\ &= (-{}^{n}m', n^{m'}) \end{aligned}$$

and

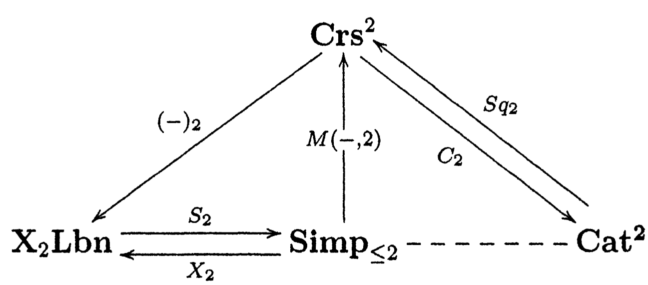
$$\begin{aligned} (m,n)^{\partial_1(m',n')} - [(m,n),(m',n')] &= (m,n)^{\delta'(m')+\delta(n')} - [(m,n),(m',n')] \\ &= ([m,m'] + m^{n'}, n^{m'} + [n,n']) \\ &\quad - ([m,m'] +^{n} m' + m^{n'}, [n,n']) \\ &= (-{}^{n}m', n^{m'}), \end{aligned}$$

the second condition of Definition 13 is satisfied. The other conditions also hold by direct checking. ■

Lemma 24 *We can also adapt the functor* $M(-,2)\colon \mathbf{Simp} \to \mathbf{Crs}^2$ *to Leibniz algebras easily, which is defined for commutative algebras in [2].*

Finally:

Corollary 25 *The functors we defined above, fit in a single diagram:*



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A GRAPH ASSOCIATED TO THE COMMUTATIVITY DEGREES OF A FINITE GROUP

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Abstract

Let G be a finite group. Then the relative commutativity degree of a subgroup H of G is the number of pairs $(h, g) \in H \times G$ such that $hg = gh$ divided by $|H||G|$. We define a graph denoted by Γ_G which is an undirected simple graph whose vertices are all subgroups of G and two distinct vertices H and K are adjacent if $d(H, G) \neq d(K, G)$. We discuss about some basic properties of the graph and state some conditions for a group G under with the graph Γ_G is complete 3,4 or 5 partite. Finally, we determine this graph for dihedral group D_{2n} where $n \geq 3$.

Keywords: Relative commutativity degree, diameter, girth, dihedral groups.

1 Introduction

Let G be a finite group and H be a subgroup of G. The relative commutativity degree is defined as the ratio

$$d(H,G) = \frac{1}{|H||G|}|\{(h,g) \in H \times G : hg = gh\}|.$$

The relative commutativity degree was introduced by the second author in 2007 (see [3]for more details). It is a generalization of commutativity degree $d(G) = d(G,G)$ which was defined by Miller in 1974 (see [10]). There are some results on the commutativity degree and related commutativity degree in a series of papers, for instance [1, 3, 4, 7, 9]. Recently, Barzegar et al (see [1]) considered the structure of all groups G in which $D(G) = \{d(H,G) : H \leq G\}$ has size 3. Later, Farrokhi and the second author [5] determined groups G such that $|D(G)| = 4$.
In this paper, we introduce a graph associated to the above relative commutativity degrees as the following:

Definition 1.1. *For a finite group G, the graph Γ_G is a graph whose vertices are all subgroups of G and two distinct vertices H and K are adjacent if $d(H,G) \neq d(K,G)$.*

In this section, we remind some basic definitions in graphs and groups which are necessary in the paper. In Section 2, some basic properties of this graph are discussed. For instance, connectivity, diameter and girth are determined. Section 3 is devoted to a consideration of group G whose associated graph Γ_G is complete 3 or 4 or 5 partite. In section 4, this graph for dihedral group of order $2n$ wherever n is even or odd are discussed.

Now, we remind some basic definitions in graphs and groups which are needed. A graph X consists of a vertex set $V(X)$ and an edge set $E(X)$, where an edge is an unordered pair of distinct vertices of X. We will usually use xy or $x - y$ to denote an edge. If $x - y$ is an edge, then we say that x adjacent to y. A simple graph is a graph with no loops and multiple edges. A complete graph is a graph in which every two vertices are adjacent. The complete graph with n vertices is denoted by K_n. A k-partite graph is a graph whose vertices can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent. The diameter of a graph X is the longest distance between two vertices in graph. The girth of a graph is the length of a shortest cycle contained in the graph. Let $k > 0$ be an integer. A k-vertex coloring of a graph X is an assignment of k colors to the vertices of X such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(X)$ of a graph, is the minimum k for which X has a k-vertex coloring. A subset of the vertices of X is called a clique, if the induced subgraph of vertices of X is a complete graph. The maximum size of a clique in a graph X is called the clique number of graph and denote by $\omega(X)$. A subset of the vertices of X is called an independent set if the induced subgraph on X has no edges. The independence number of X is the maximum size of an independent set of vertices and is denoted by $\alpha(X)$. A subset D of the vertices of X is said to be a dominating set if for every vertex outside of D there is a vertex in D and edge between them. The minimum size of a dominating set is called dominating number and denote it by $\gamma(X)$. A planer graph is a graph that can be embedded on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. A 1-planer graph is a graph that can be drown in the plane in such a way that each has at most one crossing point, where it crosses a single additional edge. Two groups G_1 and G_2 are called isoclinic if there are isomorphism α from $G_1/Z(G_1)$ to $G_2/Z(G_2)$ and β from the derived subgroup G_1' to G_2' such that if $\alpha(x_1Z(G_1)) = y_1Z(G_2)$ and $\alpha(x_2Z(G_1)) = y_2Z(G_2)$ with $x_1, x_2 \in G_1$ and $y_1, y_2 \in G_2$, then $\beta([x_1, x_2]) = [y_1, y_2]$. We say that G_1 and G_2 are isoclinic, writing briefly $G_1 \sim G_2$. A group G is said to be a nilpotent group if there exists central series of G defined as $\{1\} = G_0 \leqslant G_1 \leqslant \ldots G_n = G$ such that $G_i \trianglelefteq G$, $\frac{G_{i+1}}{G_i} \leq Z(\frac{G}{G_i})$ for $0 \leqslant i \leqslant n-1$. A group G is called Frobenius group if there are proper subgroup H and K of G such that $G = HK$ and $H \cap H^g = 1$ for every $g \in G \setminus H$.

In this paper, we assume that all graphs are simple and all groups are finite. Moreover all notations and terminologist about the graphs and groups are standard (for instance see [11, 6]).

2 Basic properties of Γ_G

As we mentioned earlier in the previous section, Γ_G is an undirected simple graph. First, let us remind the following definition from [1] which plays an important role in this paper.

Definition 2.1. *Let G be a finite group. Then we denote*

$$D(G) = \{d(H,G) : H \leq G\}.$$

It is obvious that if H is the identity subgroup or generally if H is a central subgroup, then $d(H,G) = 1$. So $D(G)$ always contain an integer 1. Also, we know that $d(G,G) = d(G)$. Thus $1, d(G) \in D(G)$. We can easily see that $|D(G)| = 1$ if and only if G is an abelian group. In this case, the graph Γ_G consists of some isolated vertices and so is disconnected. One can cheek that there is not any group G with $|D(G)| = 2$ and so $|D(G)| \geq 3$, for every non-abelian group G. Now, assume that G is a finite non-abelian group and $D(G) = \{1 = d_1, d_2, d_3, \ldots, d_n\}$ and $V_i = \{H \leq G : d(H,G) = d_i\}$, for $i = 1, 2, \ldots, n$. Then the graph Γ_G is complete n-partite graph with vertex set $V = V_1 \cup V_2 \cup \ldots \cup V_n$. It is clear that for every i size of the subset V_i is at least one. So, if $S(G)$ is the set of all subgroups of G, then $|D(G)| = n \leq |S(G)|$. In the case that $n = |S(G)|$, we should have $|V_i| = 1$ for every $1 \leq i \leq n$ and so that graph Γ_G is a complete graph K_n. Therefor, diam$(\Gamma_G) = 1$ in this case. Furthermore, independence, chromatic and dominating numbers are $1, n$ and 1, respectively. Thus, from now on we may consider the case that G is finite non-abelian group and $|D(G)| < |S(G)|$.
In the following theorem, we determine diameter of Γ_G, precisely.

Theorem 2.2. *Let G be a non-abelian finite group. Then*

$$\text{diam}(\Gamma_G) = \begin{cases} 1 & \textit{if } |D(G)| = |S(G)| \\ 2 & \textit{Otherwise} \end{cases}$$

Proof. Since G is non-abelian, $|D(G)| \geq 3$ and assume that $V(\Gamma_G) = V_1 \cup V_2 \cup \ldots \cup V_n$, where $n = |D(G)| = |\{1 = d_1, d_2, d_3, \ldots, d_n\}| \geq 3$ and

$V_i = \{H \leq G : d(H,G) = d_i\}$. If $S(G) = \{H : H \leq G\}$, and $n = |S(G)|$, then as we mentioned before, $|V_1| = |V_2| = \ldots |V_n| = 1$ and $n = |V(\Gamma_G)|$. Hence Γ_G is a complete graph K_n and so $\text{diam}(\Gamma_G) = 1$. Now, assume that $n < |S(G)|$. Then there exists a subset V_i of the vertex set such that $|V_i| \geq 2$. If $H_1, H_2 \in V_i$ then H_1 and H_2 are not adjacent. Take a subgroup H_3 from any subset V_j, $i \neq j$, then we will have a path $H_1 - H_3 - H_2$ of length 2. On other hand, we always have $\text{diam}(\Gamma_G) \leq 2$. Hence $\text{diam}(\Gamma_G) = 2$, in this case. □

One may note that if G is non- abelian then Γ_G is connected. The following theorem determines the girth of this graph.

Theorem 2.3. *Let G be a non-abelian group. Then* $\text{girth}(\Gamma_G) = 3$.

Proof. Since G is a non-abelian we have $V(\Gamma_G) = V_1 \cup V_2 \cup \ldots \cup V_n$, where $n \geq 3$. Thus take three subgroups H_1, H_2, H_3 from V_1, V_2, V_3, respectively. Then we will have a triangle $H_1 - H_2 - H_3 - H_1$ and so $\text{girth}(\Gamma_G) = 3$. □

Now, we are going to find some numerical invariants of graph Γ_G. First, assume that G is a non-abelian, $D(G) = \{1 = d_1, d_2, d_3, \ldots, d_n\}$, $V(\Gamma_G) = V_1 \cup V_2 \cup \ldots \cup V_n$ and $|V_i| = m_i$, where $1 \leq i \leq n$ and $n \geq 3$. It is clear that $|V(\Gamma_G)| = m_1 + m_2 + \ldots + m_n$. Thus we may state the following results.

Theorem 2.4. *Let G be a non-abelian group. Then by the above notations we have*

(i) $\alpha(\Gamma_G) = max\{m_1, m_2, \ldots, m_n\}$

(ii) $\gamma(\Gamma_G) = min\{n, m_1, m_2, \ldots, m_n\}$

(iii) $\omega(\Gamma_G) = \chi(\Gamma_G) = n$

Proof.

(i) It is obvious that every subset V_i is an independent set and there is no independent set having elements form at least two subsets V_i and V_j. Hence the size of the largest subset V_i is an independence number and the result follows.

(ii) It is clear that every subset V_i is a dominating set. Moreover, if we take one subgroup H_i of V_i, for $1 \leq i \leq n$, then the set $D = \{H_1, H_2, \ldots, H_n\}$ is also dominating set. Hence the dominating number is the minimum size of subsets $V_1, V_2, \ldots, V_n$ and D.

(iii) It is very straightforward. □

Theorem 2.5. *Let G_1 and G_2 be two isoclinic groups. Then they have the same chromatic and clique numbers.*

Proof. By Lemma 4.4 in [1], we have $D(G_1) = D(G_2)$ and so the proof follows immediately. □

The end of this section deals with planarity and 1-planarity of Γ_G under with the graph Γ_G is complete 3,4 or 5 partite.

Theorem 2.6. *Suppose that G is a non-abelian, $D(G)$=$\{1$=$d_1, d_2, d_3, \ldots, d_n\}$, $V(\Gamma_G) = V_1 \cup V_2 \cup \ldots \cup V_n$ and $|V_i| = m_i$, where $1 \leq i \leq n$.*

(i) *If $n = 3$ then Γ_G is planar if and only if $m_1 \geq 3$, $m_2 = m_3 = 1$ or $m_1, m_2, m_3 \leq 2$.*

(ii) *If $n = 4$ then Γ_G then Γ_G is planar if and only if $m_1 = 1$ or 2, $m_2 = m_3 = m_4 = 1$.*

Proof. It follows by [8] and the fact that when a complete multi partite graphs are planar. □

By the above notations, we can consider 1-planarity as the following theorem.

Theorem 2.7. *Let G be a finite non-abelian group then by the same notations as in Theorem 2.6 we have the following:*

(i) *If $n = 3$ then Γ_G is 1-planar if and only if $1 \leq m_1 \leq 6$, $m_2 = m_3 = 1$ or $2 \leq m_1 \leq 6$, $m_2 = 2, m_3 = 1$ or $2 \leq m_1 \leq 4$ and $m_2 = 3, m_3 = 1$.*

(ii) *If $n = 4$ then Γ_G is 1-planar if and only if $1 \leq m_1 \leq 6$, $m_2 = m_3 = m_4 = 1$ or $2 \leq m_1 \leq 3$, $m_2 = 2, m_3 = m_4 = 1$ or $m_1 = m_2 = m_3 = 2$, $1 \leq m_4 \leq 2$.*

(iii) *If* $n = 5$ *then* Γ_G *is 1-planar if and only if* $1 \leq m_1 \leq 2$ *and* $m_2 = m_3 = m_4 = m_5 = 1$ *or* $m_1 = m_2 = 2, m_3 = m_4 = m_5 = 1$.

(iv) *If* $n = 6$ *then* Γ_G *is 1-planar if and only if* $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$

(v) *If* $m \geq 7$, *then* Γ_G *contains* K_7 *as a subgraph, hence it cannot be 1-planar.*

Proof. It follows by [2] and the similar method as the proof of Theorem 2.6. □

Example 2.8. *Let* D_{2p} *be dihedral group of order* $2p$, *where* p *is a prime number, then* $\Gamma_{D_{2p}}$ *is a planar graph. Because* $\Gamma_{D_{2p}}$ *is complete* 3*-partite graph by a result that we will state later in section* 4 *the number of vertexes in every part are* $1, 1$ *and* $p + 1$. *So* $\Gamma_{D_{2p}}$ *is a planar graph. We can also easily see that* $\Gamma_{D_{2p}}$ *is 1-planner, if* p *is* $2, 3$ *and* 5, *by Theorem* 2.7.

3 Complete 3,4 and 5-partite graphs

In this section, we study the structure of some groups such that their related graphs are complete 3,4 or 5-partite graphs. As we mentioned earlier in section 1, it is not possible that Γ_G is complete or complete bipartite graphs, when G is not abelian.

Theorem 3.1. *Let* G *be a finite non-nilpotent group such that* $G/Z(G)$ *is a non-cyclic group of order* pq, *where* p *and* q *are distinct primes* $p > q$. *Then* Γ_G *is a complete* 3*-partite graph.*

Proof. Suppose that G is a finite non-nilpotent group such that $|G/Z(G)| = pq$, $p > q$. Let H be a non-centeral subgroup of G then $d(HZ(G), G) = d(H, G)$ and we may assume that $Z(G) \subseteq H$. Thus $H/Z(G) \cong Z_p$ or Z_q. In the first case, if $H/Z(G) \cong Z_p$ then for $h \notin Z(G)$, $|C_G(h)/Z(G)| = p$ therefor $|h^G| = q$. Thus

$$\begin{aligned} d(H,G) &= \frac{1}{|H|}\Big(\sum_{h\in Z(G)} \frac{1}{|h^G|} + \sum_{h\in H\setminus Z(G)} \frac{1}{|h^G|}\Big) \\ &= \frac{1}{|H|}(|Z(G)| + (p-1)|Z(G)|\frac{1}{q}) = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} \end{aligned}$$

In the second case, if $H/Z(G) \cong Z_q$ then we have

$$\begin{aligned} d(H,G) &= \frac{1}{|H|}\Big(\sum_{h\in Z(G)} \frac{1}{|h^G|} + \sum_{h\in H\setminus Z(G)} \frac{1}{|h^G|}\Big) \\ &= \frac{1}{|H|}(|Z(G)| + (q-1)|Z(G)|\frac{1}{p}) = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} \end{aligned}$$

On the other hand, $\overline{G} = \frac{G}{Z(G)}$ is a Frobenius group with Frobenius kernel, $\overline{K} = \frac{K}{Z(G)} \cong Z_p$ and complement $\overline{H} = \frac{H}{Z(G)} \cong Z_q$. Thus $|x^G| = |h^G| = p$ and we have

$$\begin{aligned} d(G) &= \frac{1}{|G|}\Big(\sum_{x\in Z(G)} \frac{1}{|x^G|} + \sum_{x\in K\setminus Z(G)} \frac{1}{|x^G|} + \sum_{x\in G\setminus K} \frac{1}{|x^G|}\Big) \\ &= \frac{1}{|G|}(|Z(G)| + (p-1)|Z(G)|\frac{1}{p} + (pq-p)|Z(G)|\frac{1}{p}) \\ &= \frac{1}{p} + \frac{1}{q^2} - \frac{1}{pq^2} \end{aligned}$$

Therefore $D(G) = \{1, \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}, \frac{1}{p} + \frac{1}{q^2} - \frac{1}{pq^2}\}$ and so $|D(G)| = 3$. Hence the proof is completed. □

Theorem 3.2. *Let G be a finite group such that $|G/Z(G)| = p^3$.*

(i) *If G has no maximal abelian subgroup, then Γ_G is a complete 4-partite graph.*

(ii) *If M is a maximal abelian subgroup of G, then Γ_G is a complete 5-partite graph.*

Proof.

(i) If G has not maximal abelian subgroup, then for every $x \in G\setminus Z(G)$, we have $|C_G(X)| = p|Z(G)|$ and if H is a subgroup of $Z(G)$ such that $[H : Z(G)] = p^i$, then

$$\begin{aligned} d(H,G) &= \frac{1}{|H|}\Big(\sum_{h\in Z(G)} \frac{1}{|h^G|} + \sum_{h\in H\setminus Z(G)} \frac{1}{|h^G|}\Big) \\ &= \frac{1}{|H|}(|Z(G)| + (|H| - |Z(G)|)\frac{1}{p^2}) = \frac{1}{p^i}(1 + (p^i - 1)\frac{1}{p^2}). \end{aligned}$$

Thus $D(G) = \{1, \frac{p^2+P-1}{p^3}, \frac{2p^2-1}{p^4}, \frac{p^3+p^2-1}{p^5}\}$ which implies that Γ_G is a complete 4-partite graph.

(ii) Assume that M is a maximal abelian subgroup of G. Suppose that H is a subgroup of $Z(G)$, then we have the following cases:
(a) If $H = Z(G)$ then $d(H, G) = 1$.
(b) If $[H : Z(G)] = p, H \nsubseteq M$ then $d(H, G) = \frac{p^2+p-1}{p^3}$.
(c) If $[H : Z(G)] = p, H \subseteq M$ then $d(H, G) = \frac{2p-1}{p^2}$.
(d) If $H = M$ then $d(H, G) = \frac{p^2+p-1}{p^3}$.
(e) If $[H : Z(G)] = p^2, H \neq M$ then $|H \cap M| = p|Z(G)|$, then

$$d(H, G) = \frac{|H||G| + (|H \cap M| - |Z(G)|)p^2|Z(G)| + (|H| - |H \cap M|)p|Z(G)|}{|H||G|}$$
$$= \frac{3p-2}{p^3}.$$

(f) If $H = G$, then

$$d(H, G) = \frac{|H||G| + (|M| - |Z(G)|)p^2|Z(G)| + (|G| - |M|)p|Z(G)|}{|G|^2}$$
$$= \frac{2p^2-1}{p^4}.$$

Therefor we have $D(G) = \{1, \frac{2p-1}{p^2}, \frac{p^2+p-1}{p^3}, \frac{3p-2}{p^3}, \frac{2p^2-1}{p^4}\}$ and so Γ_G is a complete 5-partite graph, as required. □

4 Special case $G = D_{2n}$

The last section of the paper is devoted to a consideration of Γ_G, when G is a dihedral group of order $2n$. First, we remind the following lemma from [1]

Lemma 4.1. *Let $G = D_{2n}$ be a dihedral group of order $2n$, then*

$$|D(G)| = \begin{cases} 2\tau(n) - 1 & n \text{ is an odd number} \\ 2r\tau(m) - 1 & n = 2^r m,\ m \text{ is an odd number and } r \geq 1 \end{cases}$$

where $\tau(n)$ is the number of the divisor of n.

Theorem 4.2. *Let D_{2n} be a dihedral group of order $2n$, and k be an odd prime number. Then $\Gamma_{D_{2n}}$ is a complete $2k-1$-partite graph if n is one of the forms 2^k, p^{k-1} and $2p^{k-1}$, for some odd prime p*

Proof. Assume that $\Gamma_{D_{2n}}$ is complete $2k-1$-partite graph. Then we should have $|D(D_{2n}| = 2k-1$. Thus, by Lemma 4.1 we have the following cases.

(i) $|D(D_{2n}| = 2\tau(n)-1$, when n is an odd number. In this case, we must have $2\tau(n)-1 = 2k-1$ which implies that $\tau(n) = k$. Thus if $n = p_1^{e_1}p_2^{e_2}\ldots p_s^{e_s}$, then $\tau(n) = (e_1+1)(e_2+1)\ldots(e_s+1)$. Since k is an odd prime number, so $k = (e_i+1)(e_j+1)$, for some $1 \leq i \neq j \leq s$. Hence $n = p_i^{k-1}$ or p_j^{k-1} as required.

(ii) $|D(D_{2n}| = 2r\tau(m)-1$, where $n = 2^r m$, m is an odd prime and $r \geq 1$. We have $2r\tau(m)-1 = 2k-1$ which implies that $r\tau(m) = k$. It is clear that if $r = 1$, then $\tau(m) = k$, so $m = p^{k-1}$, for some odd prime p. Therefor, $n = 2p^{k-1}$. If $r = k$, then $\tau(m) = 1$, so $m = 1$. Consequently $n = 2^k$ and the proof is completed. □

The following theorem is a direct consequence of the above theorem which states that when $\Gamma_{D_{2n}}$ is complete m-partite graph, $2 \leq m \leq 5$ or m is even number. We omit the proof.

Theorem 4.3. *Let D_{2n} be dihedral group of order $2n$, and p be an arbitrary prime number. Then*

(i) *$\Gamma_{D_{2n}}$ is complete 3-partite graph, if $n = p$ or $n = 2p$.*

(ii) *$\Gamma_{D_{2n}}$ is complete 5-partite graph, if $n = p^2$ or $n = 2p^2$.*

(iii) *$\Gamma_{D_{2n}}$ can not be complete k-partite graph if k is an even number.*

Finally, we give the following examples which confirm our results. One can observe that subgroups of D_{2n} are as the form $< a^k >$, $< a^l b >$ or $< a^k, a^l b >$, where $k|n$ and $l = 0, 1, \ldots \frac{n}{k} - 1$.

Example 4.4.

(i) *Suppose* $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1} \rangle$ *and* H_i, $1 \leq i \leq 10$ *are subgroups of* D_8 *as the following,*

$$H_1 = \{e\}, H_2 = \{1, a^2\}, H_3 = \{1, a, a^2, a^3\}, H_4 = \{1, b\}, H_5 = \{1, ab\},$$
$$H_6 = \{1, a^2 b\}, H_7 = \{1, a^3 b\}, H_8 = \{1, a^2, ab, a^3 b\}, H_9 = \{1, a^2, b, a^2 b\},$$
$$H_{10} = G.$$

Then we can see that $d(H_1, G) = d(H_2, G) = 1$, $d(H_i, G) = 12/16$ *for* $3 \leq i \leq 9$, $d(H_{10}, G) = 10/16$. *Then* Γ_{D_8} *is a complete* 3*-partite graph.*

(ii) *Let* $D_{10} = \langle a, b : a^5 = b^2 = 1, a^b = a^{-1} \rangle$ *be dihedral group of order* 10 *and* H_i, $1 \leq i \leq 8$ *be subgroups of* D_{10} *as the following*

$$H_1 = \{e\}, H_2 = \{1, a, a^2, a^3, a^2\}, H_3 = \{1, ab\}, H_4 = \{1, a^2 b\}, H_5 = \{1, a^3 b\},$$
$$H_6 = \{1, a^4 b\}, H_7 = \{1, b\}, H_8 = G.$$

It is clear that $d(H_1, G) = 1$, $d(H_i, G) = 3/5$ *for* $2 \leq i \leq 7$, $d(H_8, G) = 2/5$. *Thus* $\Gamma_{D_{10}}$ *is a complete* 3*-partite graph.*

(iii) *Let* $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ *be quaternion group of order* 8 *then all subgroups of* Q_8 *are*

$$H_1 = \{e\}, H_2 = \{1, -1, i, -i\}, H_3 = \{1, -1, j, -j\}, H_4 = \{1, -1, k, -k\},$$
$$H_5 = G.$$

Therefor $d(H_1, G) = 1$, $d(H_i, G) = 12/16$ *for* $1 \leq i \leq 5$, $d(G, G) = 10/16$. *Then* Γ_{Q_8} *is a complete* 3*-partite graph.*

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JIANG'S FUNCTION $J_{n+1}(\omega)$ IN PRIME DISTRIBUTION

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Abstract

We define that prime equation

$$f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots P_n) \quad (5)$$

are polynomials (with integer coefficients) irreducible over integers, where $P_1,\cdots,P_n$ are all prime. If Jiang's function $J_{n+1}(\omega)=0$ then (5) has finite prime solutions. If $J_{n+1}(\omega)\neq 0$ then there are infinitely many primes $P_1,\cdots,P_n$ such that $f_1,\cdots f_k$ are primes. We obtain a unite prime formula in prime distribution

$$\pi_{k+1}(N,n+1)=\left|\{P_1,\cdots,P_n\le N: f_1,\cdots,f_k \text{ are } k \text{ primes}\}\right|$$

$$=\prod_{i=1}^{k}(\deg f_i)^{-1}\times\frac{J_{n+1}(\omega)\omega^k}{n!\phi^{k+n}(\omega)}\frac{N^n}{\log^{k+n}N}(1+o(1)). \quad (8)$$

Jiang's function is accurate sieve function. Using Jiang's function we prove about 600 prime theorems [6]. Jiang's function provides proofs of the prime theorems which are simple enough to understand and accurate enough to be useful.

Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate.

Leonhard Euler

It will be another million years, at least, before we understand the primes.

Paul Erdős

Suppose that Euler totient function

$$\phi(\omega) = \prod_{2 \le P}(P-1) = \infty \quad \text{as} \quad \omega \to \infty, \tag{1}$$

where $\omega = \prod_{2 \le P} P$ is called primorial.

Suppose that $(\omega, h_i) = 1$, where $i = 1, \cdots, \phi(\omega)$. We have prime equations

$$P_1 = \omega n + 1, \cdots, P_{\phi(\omega)} = \omega n + h_{\phi(\omega)} \tag{2}$$

where $n = 0, 1, 2, \cdots$.

(2) is called infinitely many prime equations (IMPE). Every equation has infinitely many prime solutions. We have

$$\pi_{h_i} = \sum_{\substack{P_i \le N \\ P_i \equiv h_i (\bmod \omega)}} 1 = \frac{\pi(N)}{\phi(\omega)}(1 + o(1)).,\tag{3}$$

where π_{h_i} denotes the number of primes $P_i \le N$ in $P_i = \omega n + h_i$ $n = 0, 1, 2, \cdots$, $\pi(N)$ the number of primes less than or equal to N.

We replace sets of prime numbers by IMPE. (2) is the fundamental tool for proving the prime theorems in prime distribution.

Let $\omega = 30$ and $\phi(30) = 8$. From (2) we have eight prime equations

$$P_1 = 30n + 1,\ P_2 = 30n + 7,\ P_3 = 30n + 11,\ P_4 = 30n + 13,\ P_5 = 30n + 17,$$

$$P_6 = 30n + 19,\ P_7 = 30n + 23,\ P_8 = 30n + 29,\ n = 0, 1, 2, \cdots \tag{4}$$

Every equation has infinitely many prime solutions.

THEOREM. We define that prime equations

$$f_1(P_1, \cdots, P_n), \cdots, f_k(P_1, \cdots, P_n) \tag{5}$$

are polynomials (with integer coefficients) irreducible over integers, where $P_1, \cdots, P_n$ are primes. If Jiang's function $J_{n+1}(\omega) = 0$ then (5) has finite prime solutions. If $J_{n+1}(\omega) \ne 0$ then there exist infinitely many primes $P_1, \cdots, P_n$ such that each f_k is a prime.

PROOF. Firstly, we have Jiang's function [1-11]

$$J_{n+1}(\omega) = \prod_{3 \le P}[(P-1)^n - \chi(P)], \tag{6}$$

where $\chi(P)$ is called sieve constant and denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k} f_i(q_1,\cdots,q_n) \equiv 0 \pmod P, \tag{7}$$

where $q_1 = 1,\cdots,P-1,\cdots,q_n = 1,\cdots,P-1$.

$J_{n+1}(\omega)$ denotes the number of sets of $P_1,\cdots,P_n$ prime equations such that $f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots,P_n)$ are prime equations. If $J_{n+1}(\omega)=0$ then (5) has finite prime solutions. If $J_{n+1}(\omega)\neq 0$ using $\chi(P)$ we sift out from (2) prime equations which can not be represented $P_1,\cdots,P_n$, then residual prime equations of (2) are $P_1,\cdots,P_n$ prime equations such that $f_1(P_1,\cdots,P_n),\cdots,$ $f_k(P_1,\cdots,P_n)$ are prime equations. Therefore we prove that there exist infinitely many primes $P_1,\cdots,P_n$ such that $f_1(P_1,\cdots,P_n),\cdots,$ $f_k(P_1,\cdots,P_n)$ are primes.

Secondly, we have the best asymptotic formula [2,3,4,6]

$$\pi_{k+1}(N,n+1) = \left|\{P_1,\cdots,P_n \le N : f_1,\cdots,f_k \text{ are } k \text{ primes}\}\right|$$

$$= \prod_{i=1}^{k} (\deg f_i)^{-1} \times \frac{J_{n+1}(\omega)\omega^k}{n!\phi^{k+n}(\omega)} \frac{N^n}{\log^{k+n} N}(1+o(1)). \tag{8}$$

(8) is called a unite prime formula in prime distribution. Let $n=1, k=0$, $J_2(\omega)=\phi(\omega)$. From (8) we have prime number theorem

$$\pi_1(N,2) = \left|\{P_1 \le N : P_1 \text{ is prime}\}\right| = \frac{N}{\log N}(1+o(1)).. \tag{9}$$

Number theorists believe that there are infinitely many twin primes, but they do not have rigorous proof of this old conjecture by any method. All the prime theorems are conjectures except the prime number theorem, because they do not prove that prime equations have infinitely many prime solutions. We prove the following conjectures by this theorem.

Example 1. Twin primes $P, P+2$ (300BC).

From (6) and (7) we have Jiang's function

$$J_2(\omega) = \prod_{3\le P}(P-2) \ne 0.$$

Since $J_2(\omega) \ne 0$ in (2) exist infinitely many P prime equations such that $P+2$ is a prime equation. Therefore we prove that there are infinitely many primes P such that $P+2$ is a prime.

Let $\omega = 30$ and $J_2(30) = 3$. From (4) we have three P prime equations

$$P_3 = 30n+11, \quad P_5 = 30n+17, \quad P_8 = 30n+29.$$

From (8) we have the best asymptotic formula

$$\pi_2(N,2) = \left|\{P \le N : P+2 \text{ prime}\}\right| = \frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+o(1))$$

$$= 2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)).$$

In 1996 we proved twin primes conjecture [1]

Remark. $J_2(\omega)$ denotes the number of P prime equations, $\frac{\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+o(1))$ the number of solutions of primes for every P prime equation.

Example 2. Even Goldbach's conjecture $N = P_1 + P_2$. Every even number $N \ge 6$ is the sum of two primes.

From (6) and (7) we have Jiang's function

$$J_2(\omega) = \prod_{3\le P}(P-2)\prod_{P|N}\frac{P-1}{P-2} \ne 0.$$

Since $J_2(\omega) \ne 0$ as $N \to \infty$ in (2) exist infinitely many P_1 prime equations such that $N - P_1$ is a prime equation. Therefore we prove that every even number $N \ge 6$ is the sum of two primes.

From (8) we have the best asymptotic formula

$$\pi_2(N,2) = \left|\{P_1 \le N, N-P_1 \text{ prime}\}\right| = \frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+o(1)).$$

$$= 2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\prod_{P|N}\frac{P-1}{P-2}\frac{N}{\log^2 N}(1+o(1)).$$

In 1996 we proved even Goldbach's conjecture [1]

Example 3. Prime equations $P, P+2, P+6$.

From (6) and (7) we have Jiang's function

$$J_2(\omega) = \prod_{5\le P}(P-3) \ne 0,$$

$J_2(\omega)$ is denotes the number of P prime equations such that $P+2$ and $P+6$ are prime equations. Since $J_2(\omega) \ne 0$ in (2) exist infinitely many P prime equations such that $P+2$ and $P+6$ are prime equations. Therefore we prove that there are infinitely many primes P such that $P+2$ and $P+6$ are primes.

Let $\omega = 30$, $J_2(30) = 2$. From (4) we have two P prime equations

$$P_3 = 30n+11, \quad P_5 = 30n+17.$$

From (8) we have the best asymptotic formula

$$\pi_3(N,2) = \left|\{P \le N : P+2, P+6 \text{ are primes}\}\right| = \frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+o(1)).$$

Example 4. Odd Goldbach's conjecture $N = P_1 + P_2 + P_3$. Every odd number $N \ge 9$ is the sum of three primes.

From (6) and (7) we have Jiang's function

$$J_3(\omega) = \prod_{3\le P}\left(P^2-3P+3\right)\prod_{P|N}\left(1-\frac{1}{P^2-3P+3}\right) \ne 0.$$

Since $J_3(\omega) \ne 0$ as $N \to \infty$ in (2) exist infinitely many pairs of P_1 and P_2 prime equations such that $N-P_1-P_2$ is a prime equation. Therefore we prove that every odd number $N \ge 9$ is the sum of three primes.

From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\{P_1,P_2\le N: N-P_1-P_2 \text{ prime}\}\right|=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

$$=\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\prod_{P|N}\left(1-\frac{1}{P^3-3P+3}\right)\frac{N^2}{\log^3 N}(1+o(1)).$$

Example 5. Prime equation $P_3=P_1P_2+2$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left(P^2-3P+2\right)\ne 0$$

$J_3(\omega)$ denotes the number of pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Since $J_3(\omega)\ne 0$ in (2) exist infinitely many pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\{P_1,P_2\le N: P_1P_2+2 \text{ prime}\}\right|=\frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Note. $\deg(P_1P_2)=2$.

Example 6 [12]. Prime equation $P_3=P_1^3+2P_2^3$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left[(P-1)^2-\chi(P)\right]\ne 0,$$

where $\chi(P)=3(P-1)$ if $2^{\frac{P-1}{3}}\equiv 1(\bmod P)$; $\chi(P)=0$ if $2^{\frac{P-1}{3}}\not\equiv 1(\bmod P)$;

$\chi(P)=P-1$ otherwise.

Since $J_3(\omega)\ne 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.

From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\{P_1,P_2\le N:P_1^3+2P_2^3\text{ prime}\}\right|=\frac{J_3(\omega)\omega}{6\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Example 7 [13]. Prime equation $P_3=P_1^4+(P_2+1)^2$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left[(P-1)^2-\chi(P)\right]\ne 0$$

where $\chi(P)=2(P-1)$ if $P\equiv 1(\bmod 4)$; $\chi(P)=2(P-3)$ if $P\equiv 1(\bmod 8)$; $\chi(P)=0$ otherwise.

Since $J_3(\omega)\ne 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\{P_1,P_2\le N:P_3\text{ prime}\}\right|=\frac{J_3(\omega)\omega}{8\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Example 8 [14-20]. Arithmetic progressions consisting only of primes. We define the arithmetic progressions of length k.

$$P_1,P_2=P_1+d,P_3=P_1+2d,\cdots,P_k=P_1+(k-1)d,(P_1,d)=1. \qquad (10)$$

From (8) we have the best asymptotic formula

$$\pi_2(N,2)=\left|\{P_1\le N:P_1,P_1+d,\cdots,P_1+(k-1)d\text{ are primes}\}\right|$$

$$=\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}(1+o(1))..$$

If $J_2(\omega)=0$ then (10) has finite prime solutions. If $J_2(\omega)\ne 0$ then there are infinitely many primes P_1 such that $P_2,\cdots,P_k$ are primes.
To eliminate d from (10) we have

$$P_3=2P_2-P_1,\quad P_j=(j-1)P_2-(j-2)P_1,3\le j\le k.$$

From (6) and (7) we have Jiang's function

$$J_3(\omega) = \prod_{3 \le P < k}(P-1)\prod_{k \le P}(P-1)(P-k+1) \ne 0$$

Since $J_3(\omega) \ne 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that $P_3, \cdots, P_k$ are prime equations. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that $P_3, \cdots, P_k$ are primes.

From (8) we have the best asymptotic formula

$$\pi_{k-1}(N,3) = \left|\{P_1, P_2 \le N : (j-1)P_2 - (j-2)P_1 \text{ prime}, 3 \le j \le k\}\right|$$

$$= \frac{J_3(\omega)\omega^{k-2}}{2\phi^k(\omega)} \frac{N^2}{\log^k N}(1+o(1))$$

$$= \frac{1}{2}\prod_{2 \le P < k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k \le P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)).$$

Example 9. It is a well-known conjecture that one of $P, P+2, P+2^2$ is always divisible by 3. To generalize above to the $k-$primes, we prove the following conjectures. Let n be a square-free even number.

1. $P, P+n, P+n^2$,

where $3|(n+1)$.

From (6) and (7) we have $J_2(3) = 0$, hence one of $P, P+n, P+n^2$ is always divisible by 3.

2. $P, P+n, P+n^2, \cdots, P+n^4$,

where $5|(n+b), b = 2,3$.

From (6) and (7) we have $J_2(5) = 0$, hence one of $P, P+n, P+n^2, \cdots, P+n^4$ is always divisible by 5.

3. $P, P+n, P+n^2, \cdots, P+n^6$,

where $7|(n+b), b = 2,4$.

From (6) and (7) we have $J_2(7) = 0$, hence one of $P, P+n, P+n^2, \cdots, P+n^6$ is always divisible by 7.

4. $P, P+n, P+n^2, \cdots, P+n^{10}$,

where $11|(n+b), b=3,4,5,9$.

From (6) and (7) we have $J_2(11)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{10}$ is always divisible by 11.

5. $P, P+n, P+n^2, \cdots, P+n^{12}$,

where $13|(n+b), b=2,6,7,11$.

From (6) and (7) we have $J_2(13)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{12}$ is always divisible by 13.

6. $P, P+n, P+n^2, \cdots, P+n^{16}$,

where $17|(n+b), b=3,5,6,7,10,11,12,14,15$.

From (6) and (7) we have $J_2(17)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{16}$ is always divisible by 17.

7. $P, P+n, P+n^2, \cdots, P+n^{18}$,

where $19|(n+b), b=4,5,6,9,16.17$.

From (6) and (7) we have $J_2(19)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{18}$ is always divisible by 19.

Example 10. Let n be an even number.

1. $P, P+n^i, i=1,3,5,\cdots,2k+1$,

From (6) and (7) we have $J_2(\omega)\neq 0$. Therefore we prove that there exist infinitely many primes P such that $P, P+n^i$ are primes for any k.

2. $P, P+n^i, i=2,4,6,\cdots,2k$.

From (6) and (7) we have $J_2(\omega)\neq 0$. Therefore we prove that there exist infinitely many primes P such that $P, P+n^i$ are primes for any k.

Example 11. Prime equation $2P_2=P_1+P_3$

From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}(P^2-3P+2)\neq 0.$$

Since $J_3(\omega)\neq 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that P_3 is prime equations. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.

From (8) we have the best asymptotic formula

$$\pi_2(N,3) = \left|\{P_1, P_2 \leq N : P_3 \text{ prime}\}\right| = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+o(1)).$$

In the same way we can prove $2P_2^2 = P_3 + P_1$ which has the same Jiang's function.

Jiang's function is accurate sieve function. Using it we can prove any irreducible prime equations in prime distribution. There are infinitely many twin primes but we do not have rigorous proof of this old conjecture by any method [20]. As strong as the numerical evidence may be, we still do not even know whether there are infinitely many pairs of twin primes [21]. All the prime theorems are conjectures except the prime number theorem, because they do not prove the simplest twin primes. They conjecture that the prime distribution is randomness [12-25], because they do not understand theory of prime numbers.

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SANTILLI'S ISOMATHEMATICS; SOME RECENT DEVELOPMENTS

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Abstract

R. M. Santilli discovered new realizations of the abstract axioms of numeric fields with characteristic zero, based on an axiom-preserving generalization of conventional associative product and consequential positive-definite generalization of the multiplicative unit, today known as Santilli isonumbers [1], and the resulting novel numeric fields are known as Santilli isofields. This mathematics is exactly applicable to the Interior Dynamical Systems. Isofields stimulated a corresponding maximum generalization of all of 20th century mathematics and its application to mechanics, today known as Santilli isomathematics and isomechanics, respectively, which is used for the representation of extended-deformable particles moving within physical media under Hamiltonian as well as contact non-Hamiltoian interactions. Second realization of the abstract axioms of a numeric field, this time with arbitrary (non-singular) negative definite generalized unit and related multiplication, today known as Santilli isodual isonumber[1] that have stimulated a second covering of 20th century mathematics and mechanics known as Santilli isodual isomathematics and isodual isomechanics. The latter methods are used for the classical as well as operator form of antimatter in full democracy with the study of matter. In this paper, starting with the concept of 'isotopy' as defined by R.H.Bruck [2] and appropriately applied by Santilli to define isofields, we apply it to the fields of non-zero characteristic resulting into Iso-Galois fields [57]. Such finite fields have immense applications, especially in Cryptography and bio-sciences. Isopermutation groups [58] is a natural outcome of application of 'isotopy' to permutation groups. Isopermutation group define a clear cut distinction between automorphic groups and isotopic groups.

1 Introduction

It was Enrico Fermi, [3] beginning of chapter VI, p.111 said "..... there are some doubts as to whether the usual concepts of geometry hold for such small region of space." His inspiring doubts on the exact validity of quantum mechanics for the nuclear structure led to the genesis of the whole new kind of generalized mathematics, called isomathematics and generalized mechanics, called as Hadronic mechanics.

The founders of analytic mechanics, such as Lagrange, Hamilton [4] and others classified dynamical systems in to two kinds. First one is the 'Exterior Dynamical system' and the second one is the more complex but generalized 'Interior Dynamical system'.

However, over a period of time the the above distinction was abandoned preventing the identification of limitations of the prevailing mathematical and physical theories. One can easily notice that *Lie's Theory* is exactly applicable to the exterior dynamical systems. It was Prof. Santilli who at the Department of Mathematics of Harvard University, for the first time, drew the attention of the scientific community towards the crucial distinction between exterior and interior dynamical systems and presented insufficiencies of prevailing mathematical and physical theories by submitting the so-called *axiom-preserving, nonlinear, nonlocal, and noncanonical isotopies of Lie's theory* [5] under the name *Lie Isotopic theory.* Further generalization as

Lie-admissible theory [6, 7] was also achieved by him.

During a talk at the conference *Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980, Ruggero Maria Santilli submitted new numbers based on certain axiom preserving generalization of the multiplication, today known as *isotopic numbers or isonumbers* [1] in short. This generalization induced the so-called *isotopies* of the conventional multiplication with consequential generalization of the multiplicative unit, where the Greek word "isotopy" from the Greek word " implied the meaning "same topology" [8,9]. Subsequently, Ruggero Maria Santilli submitted a new conjugation, under the name *isoduality* which yields an additional class of numbers, today known as *isodual isonumbers* [1]

It should be quite clear that there can not be new numbers without new fields. This led Santilli to define 'Isofield' which is the first new algebraic structure defined by him. This concept of 'Isofield' further led to a plethora of new isoalgebraic structures and a whole new 'Isomathematics' which is a step further in Modern Mathematics. Subsequently, 'Isomathematics' has grown in to a huge tree with various branches like 'Isofunctional Analysis', 'Isocalculus', 'Isoalgebra', isocryptography etc.

In a nutshell, *the theory of isonumbers is at the foundation of current studies of nonlinear-nonlocal-nonhamiltonian systems in nuclear, particle and statistical physics, superconductivity and other fields.*

1.1 Origin of Isonumbers

The concept of 'Isotopy' plays a vital role in the development of this new age mathematics ref. R.H.Bruck [2] and [22].
The first and foremost algebraic structure defined by Santilli is 'isofield'. Elements of an isofield are called as 'isonumbers'. The conversion of unit 1 to the isounit $\hat{1}$ is of paramount importance for further development of 'Isomathematics'.
The reader should be aware that there are various definitions of "fields" in the mathematical literature [23], [24], [25] and [17] with stronger or weaker conditions depending on the given situation. Often "fields" are assumed to be associative under the multiplication.

i.e. $a \times (b \times c) = (a \times b) \times c \quad \forall a, b, c \in F$.
We formally define an isofield [26], [27] as follows.

Definition 1.1. *Given a "field" F , here defined as a ring with with elements $a, b, c\ldots$, sum $a + b$, multiplication ab, which is commutative and associative under the operation of conventional addition $+$ and (generally nonassociative but) alternative under the operation of conventional multiplication $\times$ and respective units 0 and 1,* ***"Santilli's isofields"*** *are rings of elements $\hat{a} = a\hat{1}$ where a are elements of F and $\hat{1} = T^{-1}$ is a positive-definite $n \times n$ matrix generally outside F equipped with the same sum $\hat{a} + \hat{b}$ of F with related additive unit $\hat{0} = 0$ and a new multiplication $\hat{a} * \hat{b} = \hat{a}T\hat{b}$, under which $\hat{1} = T^{-1}$ is the new left and right unit of F in which case $\hat{F}$ satisfies all axioms of the original field.*

T is called the **isoelement**. In the above definitions we have assumed "fields" to be alternative, i.e.

$$a \times (b \times b) = (a \times b) \times b, \quad (a \times a) \times b = a \times (a \times b) \quad \forall a, b \in F.$$

Thus, "isofields" as per above definition are not in general isoassociative, i.e. they generally violate the isoassociative law of the multiplication, i.e.

$$\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c} \quad \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}$$

but rather satisfy isoalternative laws.
The specific need to generalize the definition of "number" to 'real numbers', complex numbers , 'quaternions' and 'octonians' suggested the above definition. The resulting new numbers are 'isoreal numbers', isocomplex numbers, 'isoquaternions' and 'isooctonians' respectively, where 'isooctonians' are alternative but not associative.
The 'isofields' $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times})$ are given by elements $\hat{a}, \hat{b}, \hat{c}\ldots$ characterized by one-to-one and invertible maps $a \to \hat{a}$ of the original element $a \in F$ equipped with two operations $(+, \hat{\times})$, the conventional addition $+$ of F and a new multiplication $\hat{\times}$ called *"isomultiplication"* with corresponding conventional additive unit 0 and a generalized multiplicative unit $\hat{1}$, called

"multiplicative isounit" under which all the axioms of the original field F are preserved.

Santilli has shown that the transition from exterior dynamical system to interior dynamical system can be effectively represented via the isotopy of conventional multiplication of numbers a and b from its simple possible associative form $a \times b$ in to the isotopic multiplication, or **isomultiplication** for short, as introduced in [8].
The lifting of the product $ab = a \times b$ of conventional numbers in to the form

$$a \hat{\times} b := a \times T \times b \tag{1}$$

denoted by $\hat{\times} = \times T \times$, where T is a fixed invertible quantity for all possible a, b called *isotopic element.*
This isomultiplication then lifts the conventional unit 1 defined by $1 \times a = a \times 1 = a$ to the *multiplicative isounit* $\hat{1}$ defined by

$$\hat{1} \hat{\times} a = a \hat{\times} \hat{1} = a, \text{where} \quad \hat{1} = T^{-1} \tag{2}$$

Under the condition that $\hat{1}$ preserves all the axioms of 1 the lifting $1 \to \hat{1}$ is an isotopy, i.e. the conventional unit 1 and the iso unit $\hat{1}$ (as well as the conventional product $a \times b$ and its isotopic form $a \hat{\times} b$) have the same basic axioms and coincide at the abstract level by conception. The isounit $\hat{1}$ is so chosen that it follows the axioms of the unit 1 namely; boundedness, smoothness, nowhere degeneracy, hermiticity and positive-definiteness. This ensures that the lifting $1 \to \hat{1}$ is an *isotopy* and conventional unit 1 and the isounit $\hat{1}$ coincide at the abstract level of conception. Thus, the isonumbers are the generalization of the conventional numbers characterized by the isounit and the isoproduct as defined above.
The liftings $a \to \hat{a}$, and $\times \to \hat{\times}$ can be used jointly or individually.
It is important to note that unlike isotopy of multiplication $\times \longrightarrow \hat{\times}$, the lifting of the addition $+ \longrightarrow \hat{+}$ implies general loss of left and right distributive laws. Hence the study of such a lifting is the question of independent mathematical investigation.
The first generalization was introduced by Prof. Santilli when he generalized the real, complex and quaternion numbers [26], [27] based on the lifting

of the unit 1 into isounit $\hat{1}$ as defined above. Resulting numbers are called *isorealnumbers*, *isocomplex numbers* and *isoquaternion numbers.*
In fact, this lifting leads to a variety of algebraic structures which are often used in physics. The following flowchart is self explanatory.
Isonumbers $\longrightarrow$ Isofields $\longrightarrow$ Isospaces $\longrightarrow$ Isotransformations $\longrightarrow$ Isoalgebras $\longrightarrow$ Isogroups $\longrightarrow$ Isosymmetries $\longrightarrow$ Isorepresentations $\longrightarrow$ Isogeometries etc.

The isounit is generally assumed to be outside the original field with all the possible compatible conditions imposed on it. For rudiments of *isomathematics* reader can refer to [1, 6, 7, 28].
The lifting of unit I to isounit $\hat{I}$ may be represented as,
$I \to \hat{I}(t, r, \dot{r}, p, T, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots)$. where t is time, r is the position vector, p is the momentum vector, ψ is the wave function and $\psi^\dagger$ are the corresponding partial differentials. The positive definiteness of the isounit $\hat{I}$ is assured by,
$\hat{I}(t, r, \dot{r}, p, T, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots) = \frac{1}{T} > 0$ where T is called the *isotopic element*, a positive definite quantity. The isonumbers are generated as, $\hat{n} = n \times \hat{I}$, $n = 0, 1, 2, 3, \ldots$.

Isofields are of two types, **isofield of first kind**; wherein the isounit does not belong to the original field, and **isofield of second kind**; wherein the isounit belongs to the original field. The elements of the isofield are called as **isonumbers**. This leads to number of new terms and parallel developments of conventional mathematics.

We mention two important propositions by Santilli [10].

Proposition 1.1. *The necessary and sufficient condition for the lifting (where the multiplication is lifted but elements are not)* $F(a, +, \times) \longrightarrow (\hat{F}, +, \hat{\times}), \hat{\times} = \times\hat{T}\times, \hat{1} = \hat{T}^{-1}$ *to be an isotopy (that is for* $\hat{F}$ *to verify all axioms of the original field* F*) is that* $\hat{T}$ *is a non-null element of the original field* F*.*

e.g. We can start with the field of real numbers $\Re$ and construct an isotopic field $\hat{\Re}$ with a new multiplicative identity as $\hat{1} = \frac{1}{2}$ where $\hat{T} = 2$ as the isounit. So, if $a \in \Re$ then $\hat{a} = a.\frac{1}{2}$. Thus the product of two iso numbers $\hat{a}$ and $\hat{b}$ will be $\hat{a}\hat{\times}\hat{b} = \frac{a}{2}.2.\frac{b}{2} = \frac{ab}{2} = \widehat{ab}$.

Proposition 1.2. *The lifting (where both the multiplication and the elements are lifted)*
$F(a,+,\times) \longrightarrow (\hat{F},+,\hat{\times}), \hat{a} = a \times \hat{1}, \hat{\times} = \times\hat{T}\times, \hat{1} = \hat{T}^{-1}$ *constitutes an isotopy even when the multiplicative isounit* $\hat{1}$ *is not an element of the original field.*

The following three theorems [22] directly follow from the definition of isofield.

Proposition 1.3. *If* $(F,+,\times)$ *is a field and* $(\hat{F},\hat{+},\hat{\times})$ *is the corresponding isofield such that the isounit* $\hat{I} \in F$ *then* $(F,+,\times) \cong (\hat{F},\hat{+},\hat{\times})$.

Clearly, the map $x \longmapsto \hat{x}$ is an isomorphism.

Proposition 1.4. *If* $(F,+,\times)$ *is a field and* $(\hat{F},\hat{+},\hat{\times})$ *is the corresponding isofield such that the isounit* $\hat{I} \notin F$ *then* $(F,+,\times)$ *is isotopic to* $(\hat{F},\hat{+},\hat{\times})$.

Proposition 1.5. *Isofield corresponding to a non-commutative field is isotopic to the original field.*

The noncommutative ring of Quaternions is an example of this type.
First we explore the very basics of Isomathematics as formulated by Santilli [1] and [21]. The concept of 'Isotopy' plays a vital role in the development of this new age mathematics. Starting with Isotopy of groupoids we develop the study of Isotopy of quasi groups and loops via Partial Planes, Projective planes, 3-nets and multiplicative 3-nets.

2 From Partial Plane to Loop

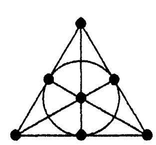
Definition 2.1. *A* ***partial plane*** *is a system consisting of a non-empty set G partitioned into two disjoint subsets(one of which may be empty), namely the point-set and the line-set together with a binary relation, called incidence, such that (i) (Disjuncture) If* x *is incident with* y *in* G *then one of* x, y *is a line of* G *and the other is a point, (ii) (Symmetry) If* x *is incident with* y *in* G *then* y *is incident with* x *in* G*, and (iii) If* x, y *are distinct elements of* G *there is at most one* z *in* G *such that* x *and* y *are both incident with* z *in* G.

Definition 2.2. *A* ***Projective plane*** *is a special kind of a partial plane G such that; (iv) if x and y are distinct points or distinct lines of G, there exists a z in G such that x and y are both incident with z in G ; (v) there exists at least one set of four distinct points of G no three of which are incident in G with the same element.*

It is easy to show that in the presence of (i)- (iv), postulate (v) is equivalent to ; (vi) there exists at least one set of four distinct lines of G no three of which are incident in G with the same element.

A *projective plane* of order n has
- $n^2 + n + 1$ points;
- $n^2 + n + 1$ lines;
- $n + 1$ points on each line;
- $n + 1$ lines through each point.

Example: The projective plane of order $n = 2$



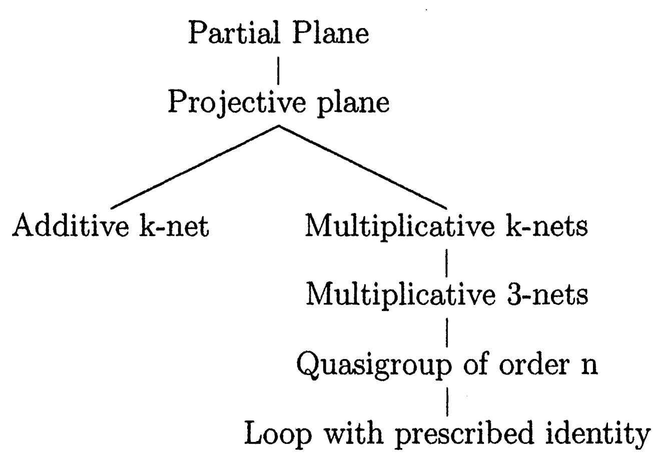
7 points
7 lines
3 points on each line
3 lines through each point

2.1 k-net

Definition 2.3. *A **k-net** is a partial plane N whose line-set has been partitioned into k disjoint classes such that (a) N has at least one point, (b) Each point of N is incident in N with exactly one line of each class, and (c) Every two lines of distinct classes in N are both incident in N with exactly one point.*

If some line of a k-net N is incident with exactly n distinct points in N, so is every line of N. The cardinal number n is called the **order of** N**.**

Since a net is a partial plane, every net may be embedded in at least one projective plane. Every projective plane contains nets and, of these, two types additive 3-net and multiplicative 3-net have special significance. The following tree diagram is self-explanatory.



3 From Geometry to Algebra

3.1 From 3-net to Loop

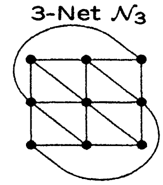
We can have an **additive 3-net** and **multiplicative 3-net** of a projective plane.

Definition 3.1. *A **Quasigroup** is a groupoid G such that, for each ordered pair $a, b \in G$, there is one and only one x such that $ax = b$ in G and one and only one y such that $ya = b$ in G.*

A quasigroup $(X, *)$ of order n determines a *3-net*, eg.

Quasigroup $(X, *)$ 3-Net $\mathcal{N}_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \longleftrightarrow$$



$n \times n$

n^2 points;
$3n$ lines;
3 parallel classes of n lines each;
n points on each line

In other words a quasigroup is groupoid whose composition table is a Latin square.

Definition 3.2. *A **loop** is a quasigroup with an identity.*

An associative loop is a group.

- *Every 3-net N of order n gives rise to a class of quasigroups* $(Q, \circ)$ *of order n* by defining one-to-one mappings $\theta(i)$ with $i = 1, 2, 3$ of Q upon the class of i-lines of N.
- Two quasigroups obtainable from the same 3-net by different choices of the set Q or of the mappings $\theta(i)$ are said to be **isotopic**.
- *For any Q, the* $\theta(i)$ *can be so chosen that* $(Q, \circ)$ *is a loop with a prescribed element e of Q as identity element.*

4 Isotopy of Groupoids

Definition 4.1. *Let $(G, \cdot)$ and $(H, \circ)$ be two groupoids. An ordered triple (α, β, γ) of one-to-one mappings α, β, γ of G upon H is called an* **isotopism** *of $(G, \cdot)$ and $(H, \circ)$. provided $(x\alpha) \circ (y\beta) = (x.y)\gamma$.*
$(G, \cdot)$ is said to be isotopic with $(H, \circ)$ or $(G, \cdot)$ is said to be an **isotope** *of $(H, \circ)$.*

Isotopy of groupoids is an equivalence relation. Every isotope of a quasigroup is a quasigroup.

4.1 Origin of Isotopy

We underline the importance of 'Isotopy' by borrowing two paragraphs from the book 'A Survey of Binary Systems' by R.H.Bruck [2].

"The concept of isotopy seems very old. In the study of Latin squares (which were known to BACHET and certainly predate Euler's problem of the 36 Officers) the concept is so natural to creep in unnoticed; and Latin squares are simply the multiplication tables of finite quasigroups." "It was consciously applied by SCHÖNHART, BAER and independently by ALBERT. ALBERT earlier had borrowed the concept from topology for application to linear algebras; in the latter theory it has virtually been forgotten except for applications to the theory of projective planes."

5 Examples on Isotopy of Groupoids

Consider the two groupoids $G = \{1, 2, 3\}$ and $G' = \{a, b, c\}$ defined by the following composition tables.

.	1	2	3
1	1	3	2
2	3	1	3
3	2	3	2

and

*	a	b	c
a	a	c	b
b	b	b	c
c	a	a	b

Then the ordered triple (α, β, γ) defined by the permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix}$ is an isotopy.
Thus $(G, .)$ and $(G, *)$ are **isotopic**.
Note that an isomorphism is just a particular case of isotopy wherein $\alpha = \beta = \gamma$. If l is the identity mapping then (α, β, l) is called a ***principal isotopy*** between the two groupoids.

5.1 Isotopy of Quasigroups

Consider the groupoids $(L, .)$ and $(L', *)$ with multiplication tables as;

.	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

and

*	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Here the ordered triple (α, β, γ) defined as $\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$ is an Isotopism. Note that L and L' are quasi groups.

5.2 Principal Isotopy of Groupoids

Consider the two groupoids G and G' defined by the following composition tables.

.	1	2	3
1	1	3	2
2	3	1	3
3	2	3	2

and

*	1	2	3
1	1	2	2
2	3	2	1
3	1	3	3

Then the ordered triple (α, β, γ) defined by the permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is a principal isotopy.

Consider the groupoids and their isotopy as defined in Example 1, we can define $\delta = \alpha\gamma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\eta = \beta\gamma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $l = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

Note that (δ, η, l) is the principal isotopy corresponding to (α, β, γ). ***In this manner every isotopy gives rise to a principal isotopy such that the isotope and the principal isotope are isomorphic.***

In general, it is sufficient to consider only the principal isotopies in view of the following Theorem.

Theorem 5.1. *If G and H are isotopic groupoids then H is isomorphic to a principal isotope of G.*

Proof. *Let (α, β, γ) be an isotopy of G on to H. Let $\delta = \alpha\gamma^{-1}$ and $\eta = \beta\gamma^{-1}$. We have $(\alpha, \beta, \gamma) = (\delta\gamma, \eta\gamma, \gamma)$. Hence there exists a groupoid K such that (δ, η, i) is a principal isotopy of G on to K and γ is an isomorphism of K on to H.*

'Necessary and sufficient conditions that a groupoid possess an isotope with identity element are that the groupoid have a right nonsingular element and a left nonsingular element' Ref.[2]p.57.

All the elements of a quasigroup are left nonsingular and right nonsingular (as every element occurs only once in every row and column). Therefore **every quasigroup is isotopic to a loop.**

- **This lifting of the multiplicative quasigroup to a loop with the prescribed identity gives rise to a multiplicative group with the 'prescribed identity' which Santilli termed as 'Isounit'. The resulting field with the multiplicative isounit is called as an Isofield [41].**

- Without loss of generality we can say that the words 'Isotopy' and 'Axioms preserving' are synonymous.

5.3 Galois fields

Finite fields were first introduced by Galois in 1830 in his proof of the un-solvability of the general quintic equation. Hence finite fields are also called as Galois fields. When Cayley invented matrices a few decades later, it was natural to investigate groups of matrices over finite fields. In fact, the groups of matrices over the finite fields have become the most important class of groups. Finite fields have vast applications in computer science, coding theory, information theory, and cryptography.

Thus, Galois fields are finite fields. Finite fields are of nonzero characteristic. Every finite field is of prime-power order, and for every power of a prime there is a unique Galois field of this order.
Santilli's isofields are defined for the fields of characteristic zero, and hence for infinite fields.
Our main purpose is to apply Santilli's ideas to the fields of non-zero characteristic and seek for further development in this direction.
In this paper we answer the open problems posed in [22],

- Can we construct finite isofields of first kind ?
- Can we construct finite isofields of second kind ?

in the affirmative.

6 Iso-Galois fields

Definition 6.1. *If F is an iso-field and F is finite, then F is called an Iso-Galois field.*

Definition 6.2. *Let F be a Galois field. If G is an Iso-Galois field of F wherein the prescribed multiplicative identity is from the field F itself, then the Iso-Galois field G is called an* ***Iso-Galois field of second kind.***

Definition 6.3. *Let F be a Galois field. If G is an Iso-Galois field of F wherein the prescribed multiplicative identity is not from the field F, then the Iso-Galois field G is called an* ***Iso-Galois field of first kind.***

If G is an Iso-Galois field constructed from the field F then we call the field F as the **trivial iso-field.**

7 Iso-Galois fields of second kind

Consider a Galois field F_8 as a set of following matrices of integers modulo 2.

$$(0)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},(1)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},(2)=\begin{pmatrix}0&1&0\\0&0&1\\1&1&0\end{pmatrix},(3)=\begin{pmatrix}0&0&1\\1&1&0\\0&1&1\end{pmatrix},$$

$$(4)=\begin{pmatrix}1&1&0\\0&1&1\\1&1&1\end{pmatrix},(5)=\begin{pmatrix}0&1&1\\1&1&1\\1&0&1\end{pmatrix},(6)=\begin{pmatrix}1&1&1\\1&0&1\\1&0&0\end{pmatrix},(7)=\begin{pmatrix}1&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

For an isofield it will be sufficient to consider the multiplication table of non-zero elements of F_8.

$\cdot$	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	1	2	3	4	5	6	7
(2)	2	3	4	5	6	7	1
(3)	3	4	5	6	7	1	2
(4)	4	5	6	7	1	2	3
(5)	5	6	7	1	2	3	4
(6)	6	7	1	2	3	4	5
(7)	7	1	2	3	4	5	6

Let us choose the iso-element $\hat{T} = (4) \equiv 4 \in F_8$ in the above composition table. Then the iso-unit element is $\hat{1} = \frac{1}{\hat{T}} = \frac{1}{4} = 4^{-1} = 5$. By using the fact that for $a \in F_8$, $\hat{a} = a \cdot \frac{1}{\hat{T}} = a \cdot 5$ we construct the corresponding isonumbers. Thus,
$\hat{2} = 2 \cdot \frac{1}{4} = 2 \cdot 5 = 6$. Similarly, $\hat{3} = 7$, $\hat{4} = 1$, $\hat{3} = 7$, $\hat{4} = 1$, $\hat{5} = 2$, $\hat{6} = 3$ and $\hat{7} = 4$.
We now construct the corresponding composition table for the isonumbers using the fact that the isomultiplication $\hat{\times}$ is defined as $\hat{a}\hat{\times}\hat{b} = \hat{a}\hat{T}\hat{b}$.
e.g. $\hat{6}\hat{\times}\hat{7} = 6 \cdot 4 \cdot 7 = 1$ using above composition table. Thus the corresponding isomultiplication table (or iso-composition table) for isonumbers will be;

$\hat{\times}$	$\widehat{(1)}$	$\widehat{(2)}$	$\widehat{(3)}$	$\widehat{(4)}$	$\widehat{(5)}$	$\widehat{(6)}$	$\widehat{(7)}$
$\widehat{(1)}$	4	5	6	7	1	2	3
$\widehat{(2)}$	5	6	7	1	2	3	4
$\widehat{(3)}$	6	7	1	2	3	4	5
$\widehat{(4)}$	7	1	2	3	4	5	6
$\widehat{(5)}$	1	2	3	4	5	6	7
$\widehat{(6)}$	2	3	4	5	6	7	1
$\widehat{(7)}$	3	4	5	6	7	1	2

Note that the numbers in the iso-composition table are isonumbers.

Remark. *1. The function $f : F_8 \to F_8$ defined by $f(x) = 5x$ is not an*

isomorphism. However,
2. The isofunction $\hat{f} : x \to \hat{x}$ is an isomorphism.

We generalize these observations in the following Theorem.

Theorem 7.1. *If F is a Galois field such that $\hat{F}$ is an Iso-Galois field of second kind, where $\hat{T} \in F$ is an isoelement and $\hat{x} = \hat{T}^{-1}x$, $x \in F$ then the function $f : F \to F$ defined by $x \mapsto \hat{T}^{-1}x$ is not an isomorphism but is an isotopism, whereas the isofunction $\hat{f} : F \to \hat{F}$ is an isomorphism.*

Proof. It is easy to verify that the function f is a translation from F to F and hence is not an isomorphism. It is an isotopism because we have prescribed a new identity and the result follows from [22].
If we consider the isofunction $\hat{f} : F \to \hat{F}$ then for $x, y \in F$, $\hat{f}(x.y) = \hat{T}^{-1}x.y$ whereas $\hat{f}(x)\hat{\times}\hat{f}(y) = \hat{x}\hat{\times}\hat{y} = \hat{T}^{-1}x.\hat{T}.\hat{T}^{-1}y = \hat{T}^{-1}x.y$. Thus $\hat{f}(x.y) = \hat{f}(x)\hat{\times}\hat{f}(y)$. □

Theorem 7.2. *If F is a Galois field of order p^m and n is the number of involutions in F then there exist $p^m - n - 1$ number of distinct Iso-Galois fields of kind two of F.*

Proof. The multiplicative group of F will obviously contain $p^m - 1$ number of non-zero elements. Every involution and its inverse will obviously give rise to the same Iso-Galois field of second kind. Therefore the total number of distinct Iso-Galois fields would be $p^m - n - 1$. □

8 Iso-Galois fields of First kind

Consider a Galois field F of order 16 represented by the polynomials $a + bx + cx^2 + dx^3$, a, b, c and d are integers modulo 2. The polynomials are generated by the powers of x using the rule $x^4 = 1 + x$.
The elements of the field are; $(0) = (0,0,0,0)$, $(1) = (1,0,0,0)$, $(2) = (0,1,0,0)$, $(3) = (0,0,1,0)$, $(4) = (0,0,0,1)$, $(5) = (1,1,0,0)$, $(6) = (0,1,1,0)$, $(7) = (0,0,1,1)$, $(8) = (1,1,0,1)$, $(9) = (1,0,1,0)$, $(10) =$

$(0,1,0,1)$, $(11) = (1,1,1,0)$, $(12) = (0,1,1,1)$, $(13) = (1,1,1,1)$, $(14) = (1,0,1,1)$, $(15) = (1,0,0,1)$ with the following composition table for multiplication;

·	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1
3	3	4	5	6	7	8	9	10	11	12	13	14	15	1	2
4	4	5	6	7	8	9	10	11	12	13	14	15	1	2	3
5	5	6	7	8	9	10	11	12	13	14	15	1	2	3	4
6	6	7	8	9	10	11	12	13	14	15	1	2	3	4	5
7	7	8	9	10	11	12	13	14	15	1	2	3	4	5	6
8	8	9	10	11	12	13	14	15	1	2	3	4	5	6	7
9	9	10	11	12	13	14	15	1	2	3	4	5	6	7	8
10	10	11	12	13	14	15	1	2	3	4	5	6	7	8	9
11	11	12	13	14	15	1	2	3	4	5	6	7	8	9	10
12	12	13	14	15	1	2	3	4	5	6	7	8	9	10	11
13	13	14	15	1	2	3	4	5	6	7	8	9	10	11	12
14	14	15	1	2	3	4	5	6	7	8	9	10	11	12	13
15	15	1	2	3	4	5	6	7	8	9	10	11	12	13	14

The set $F_1 = \{0, 1, 6, 11\}$ forms a subfield of F. We consider an element $\hat{T} = 2$ such that $\hat{T}$ does not belong to F_1 and form an Iso-Galois field of F_1. The isounit $\hat{1} = \frac{1}{\hat{T}} = \hat{T}^{-1} = 2^{-1} = 15$. The numbers of F_1 are converted to following isonumbers as $\hat{1} = 15$, $\hat{6} = 6.15 = 5$ and $\hat{11} = 11.15 = 10$. Thus the isofield is $\hat{F_1} = \{0, 15, 5, 10\}$ with the following composition table for isomultiplication.

$\hat{\times}$	15	5	10
15	15	5	10
5	5	10	15
10	10	15	5

e.g the isoproduct of 5 and 10 is given by $5.\hat{T}.10 = 5.2.10 = 15$.

Similarly, if $\hat{T} = 7$ then $\hat{1} = \frac{1}{\hat{T}} = 7^{-1} = 10$. The numbers of F_1 are converted to following isonumbers as $\hat{1} = 10$, $\hat{6} = 6.10 = 15$ and $\hat{11} = 11.10 = 5$. Thus the isofield is $\hat{F_1} = \{0, 10, 15, 5\}$ with the following composition table for isomultiplication.

$\hat{\times}$	10	15	5
10	10	15	5
15	15	5	10
5	5	10	15

Theorem 8.1. *If F is a Galois field of order p^n and F_1 is a subfield of F of order p^m such that $F \setminus F_1$ has r number of involutions, then there exist $p^n - p^m - r$ number of distinct Iso-Galois fields of first kind of F_1.*

Proof. F_1 is a subfield of F implies $m|n$. Let $n = m + r$. Then
$o(F \setminus F_1) = p^n - p^m = p^{m+r} - p^m = p^m(p^r - 1) \ldots (1)$.
Case I : If $p = 2$ then (1) $\Rightarrow o(F \setminus F_1)$ is even.
Case II ; If $p \neq 2$ then $p^r - 1$ is even and again $o(F \setminus F_1)$ is even.
Also, if $x \in F \setminus F_1$ then x^{-1} also belongs to $F \setminus F_1$. Thus, each $x \in F \setminus F_1$ gives rise to one isofield of first kind of F_1. Now, if x is an involution then $x = x^{-1}$ and hence x and x^{-1} will give rise to same isofield. If there are r number of involutions in $F \setminus F_1$ then the number of elements which give rise to distinct isofields will be $p^n - p^m - r$. Hence the result. □

9 Isotopically Isomorphic Realization

Santilli's isofields have multiplicative group which is isotopically isomorphic realization (when the isounit is from the field itself) or isotopically isomorphic representation (when the isounit is from outside the field) of the multiplicative group of the original field. For a given group G if we consider all permutations of the group elements, then there exist permutations which are the right regular representations, and permutations which are the left regular representations of G. In this case, each permutation is actually an isogroup with respect to the induced isounit and isomultiplication. These isogroups can have extensive applications in cryptography, coding theory and biological sciences wherein the codes and the keys are to be continuosly changing for the strongest security.

Isotopism of the fields [1], [10] and [12] defined by R.M. Santilli induces isotopism of the groups in the following manner.

Definition 9.1. *If G is a group then the permutation of the elements of G by $a \in G$ from the left (or right) is an isogroup $\widehat{G}$ with isomultiplication $\widehat{\times}$ defined on it. We say that $\widehat{G}$ is the* ***left-isotopically isomorphic realization*** *(or* ***right-isotopically isomorphic realization****) of the group G.*

Thus, G itself is the isotopically isomorphic realization (both left and right) of the group G where $a = e$.

Definition 9.2. *If G is a group then the set of all left-isotopically isomorphic realizations (or right-isotopically isomorphic realizations) of G is called the* ***left-isopermutation group*** *(or* ***right-isopermutation group****).*

Definition 9.3. *A group for which both left-isopermutation group and right-isopermutation group coincide is called an* ***Iso-permutation group****.*

Remark:- 1) Note that if the group G is abelian then both left-isotopically isomorphic realizations and right-isotopically isomorphic realizations of G are same. Hence G is an isopermutation group.
2) Also, the left-isopermutation group of G is the left regular representation of G which is faithful. Similarly, the set of right-isopermutation group of G is the right-regular representation of G which is faithful.

Proposition 9.1. *The isomorphic realizations of a group G consist of a class of automorphisms of the group G and the class of translations of the group G by the elements from the group G itself.*

Proof. Aut(G) forms a class of permutations of G where identity is mapped with identity. Also, left regular representations and right regular representations form a class of permutations which are isotopically isomorphic to G. □

Let us consider the composition table of the abelian group $G = \{1, a, b, c\}$ with respect to multiplication.

·	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	a	1
c	c	b	1	a

Clearly, the set of left-isotopically isomorphic realizations of $G =$ the set of right-isotopically isomorphic realizations of $G =$ left regular representation of $G =$ right regular representation of G

$$= \left\{ \begin{pmatrix} 1 & a & b & c \\ 1 & a & b & c \end{pmatrix}, \begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \end{pmatrix}, \begin{pmatrix} 1 & a & b & c \\ b & c & a & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & b & c \\ c & b & 1 & a \end{pmatrix} \right\} < S_4.$$

Let us consider a non-abelian group,
$S_3 = \{e, f, g, h, i, j\} = \{(1), (23), (12), (123), (132), (13)\}$ with the following composition table;

·	e	f	g	h	i	j
e	e	f	g	h	i	j
f	f	e	h	g	j	i
g	g	i	e	j	f	h
h	h	j	f	i	e	g
i	i	g	j	e	h	f
j	j	h	i	f	g	e

Then,
set of left-Isotopically isomorphic realizations of $G =$ left regular representation of G

$$= \left\{ \begin{pmatrix} e & f & g & h & i & j \\ e & f & g & h & i & j \end{pmatrix}, \begin{pmatrix} e & f & g & h & i & j \\ f & e & h & g & j & i \end{pmatrix}, \begin{pmatrix} e & f & g & h & i & j \\ g & i & e & j & f & h \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} e & f & g & h & i & j \\ h & j & f & i & e & g \end{pmatrix}, \begin{pmatrix} e & f & g & h & i & j \\ i & g & j & e & h & f \end{pmatrix}, \begin{pmatrix} e & f & g & h & i & j \\ j & h & i & f & g & e \end{pmatrix} \right\} < S_6.$$

Proposition 9.2. *Permutations which represent the isotopically isomorphic groups of a group G form a subgroup of the group S_G.*

Proof. left-isotopically isomorphic realizations (or right-isotopically isomorphic realizations) of a group is the left regular representation (or right regular representation) of G and hence form a subgroup of S_G. □

Proposition 9.3. *If G is an abelian group of order n then the number of isotopically isomorphic realizations of G is n.*

Proof. Each element of the group will give rise to a left translation of the elements of G which is same as the right translation. □

Proposition 9.4. *If G is a non-abelian group of order n then the number of isotopically isomorphic realizations of G is $2n-1$.*

Proof. Each element of G will give rise to distinct left and right translations of the group G. □

Proposition 9.5. *Holomorph of a group is the semidirect product of $Aut(G)$ and left-isopermutation group (or right-isopermutation group) of G.*

Proof. Let G be a group and λ and ρ be the left regular and right regular permutation representation of G defined by $g^\lambda : x \to g^{-1}x$ and $g^\rho : x \to xg$. Clearly, both λ and ρ are regular and faithful representations of G. Thus, the left regular and right regular permutation representation of G form subgroups G^λ and G^ρ of Sym G. But G^λ = left isopermutation group of G and G^ρ = right isopermutation group of G. We know that [?], Hol $G = \langle G^\lambda, Aut(G)\rangle = \langle G^\rho, Aut(G)\rangle < Sym(G)$. Hence Hol G = ⟨left isopermutation group, Aut (G)⟩ = ⟨right isopermutation group, Aut(G)⟩ $< Sym(G)$. Also, Hol$(G) = Aut(G) \ltimes G^\lambda$ and Hol$(G)= Aut(G) \ltimes G^\rho$. Hence the result.

□

Proposition 9.6. *If G is a group then the left-isopermutation group of G (right-isopermutation group of G) is the centralizer of right-isopermutation group of G (left-isopermutation group of G) in the holomorph of the group G.*

Proof. Let G be a group. Consider the left regular representation G^λ of G and the right regular representation G^ρ of G defined by $g^\lambda : x \to g^{-1}x$ and $g^\rho : x \to xg$. Then both λ and ρ are faithful representations of G. Also, the equations $C_{HolG}(G^\rho) = G^\lambda$ and $C_{HolG}(G^\lambda) = G^\rho$ hold for any group G.

□

9.1 Applications and advances

Quantum mechanics was sufficient to deal with 'Exterior Dynamical systems' which are liner, local, lagrangian and hamiltonian. The main purpose of formulating the new generalized mathematics was to deal with the insufficiencies in the modern mathematics to describe 'Interior Dynamical systems' which are intrinsically non-linear, non-local, non-hamiltonian and non-lagrangian. The axiom-preserving generalization of quantum mechanics which can also deal with non-linear, non-local non-hamiltonian and non-lagrangian systems is called the *Hadronic mechanics.* The mechanics; built specifically to deal with 'hadrons' (strongly interacting particles) ref. [21]. Prof. Santilli, in 1978 when at Harvard University, proposed 'Hadronic mechanics' under the support from U. S. Department of Energy, which was subsequently studied by number of mathematicians, theoreticians and experimentalists. Hadronic mechanics is directly universal; that is, capable of representing all possible nonlinear, nonlocal, nonhamiltonian, continuous or discrete, inhomogeneous and anisotropic systems (universality), directly in the frame of the experimenter (direct universality). In particular the hadronic mechanics has shown that quantum mechanics is completely inapplicable to the synthesis of neutron [50], as mass of the neutron is greater than the sum of the masses of proton and electron (called "mass defect") of which it is made. In this case quantum equations are completely inconsistent. Hadronic mechanics has achieved numerically exact results in the cases in which quantum mechanics results are not valid. For further details of isonumber theory we recommend refs. [51, 1, 52, 50, 53].

As far as mathematics is concerned, one of the major applications of isonumber theory is in Cryptography, ref. [54]. Cryptograms can be lifted to iso-cryptograms which render highest security for a given crypto-system.

Isonumbers, hypernumbers and their pseudo-formulations can be used effectively for the tightest security via new disciplines, *isocryptology, genocryptology, hypercryptology, pseudocryptology* etc. More complex cryptograms can be achieved using *pseudocryptograms* in which we have the additional hidden selection of addition and multiplication to the left and those to the right whose results are generally different among themselves. Yet more complex pseudocryptograms can be achieved in which the result of each individual operations of addition and multiplication is given by a set of numbers [54]. *Santillian iso-crypto systems* have maximum security due to a large variety of isounits which can be changed automatically and continuously, achieving maximum possible security needed for the modern age banking and other systems related with information technology.

Reformulations of conventional numbers to the most generalized isonumbers and subsequently to genonumbers and hypernumbers led to a vast variety of parallel developments in the conventional mathematics including hyperstructures [55] and its various branches such as 'iso-functional analysis' ref [38], iso-calculus ref [56], iso-cryptography [54] etc.
Iso-Galois fields [57], Iso-permutation groups [58, 57] have been defined by this author, which can play an important role in cryptography and other branches of mathematics where finite fields are used. Investigations are underway.

Isomathematics can also explain complex biological structures and hence has applications in Fractal geometry. Further applications in Neuroscience and Genetics can provide new insight in these disciplines.

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